

Lecture 5 Inverse Matrices

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Recap

Block multiplication If the cuts between columns of A match the cuts between rows of B, then block multiplication of AB is allowed:

$$\begin{bmatrix}
n_1 & n_2 \\
m_1 \begin{bmatrix} A_{11} & A_{12} \\
A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & \cdots \\
B_{21} & \cdots \end{bmatrix} \begin{bmatrix} n_1 & k_1 & n_2 & k_1 \\
m_1 & B_1 & H & A_{12} & B_{21} & \cdots \\
m_2 & M_{21} & B_{11} & + A_{22} & B_{21} & \cdots \end{bmatrix}.$$
(1)



Strang Sections 2.5 – Inverse Matrices

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text



Suppose A is an $n \times n$ matrix (square matrix), then A is invertible if there exists a matrix A^{-1} such that

$$AA^{-1} = I$$
 and $A^{-1}A = I$.

We can only talk about an inverse of a square matrix, but not all square matrices are invertible. We will discuss such restrictions in future lectures.

Recall: The multiplicative inverse (or reciprocal) of a nonzero number a is the number b such that ab = 1. We define the inverse of a matrix in almost the same way.

Definition

Let A be an $n \times n$ square matrix. We say A is **invertible** (or **nonsingular**) if there is a matrix B of the same size, such that identity matrix

$$AB = I_n$$
 and $BA = I_n$.

In this case, B is the **inverse** of A , and is written A^{-1} .

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

I claim $B = A^{-1}$. Check:

$$2x_1 + 4x_2 - 2x_3 = 2$$

Consider the following system: $4x_1 + 9x_2 - 3x_3 = 8$

$$4x_1 + 9x_2 - 3x_3 = 8$$

$$-2x_1 - 3x_2 + 7x_3 = 10$$

Our goal is to find x_1 , x_2 , and x_3 . In matrix form, this system is:

$$\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

$$\vec{x} = \vec{b}$$

$$\vec{dea} \qquad \vec{x} = A^{-1} \vec{b}$$

$$\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

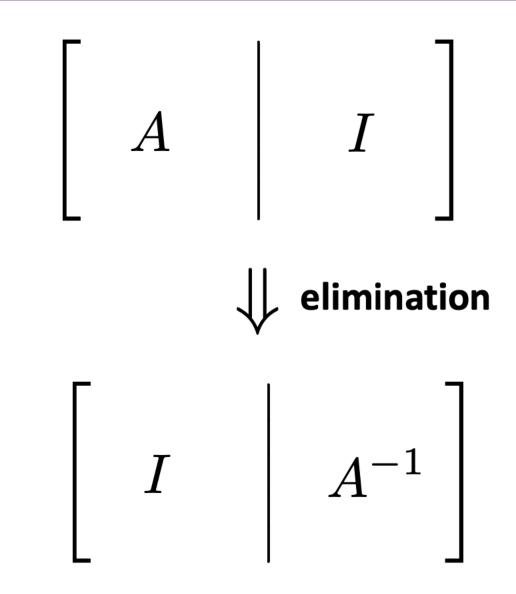
$$A \qquad \vec{x} = \vec{b}$$

$$\iff \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

$$A \qquad \vec{x} = I \qquad \vec{b}$$

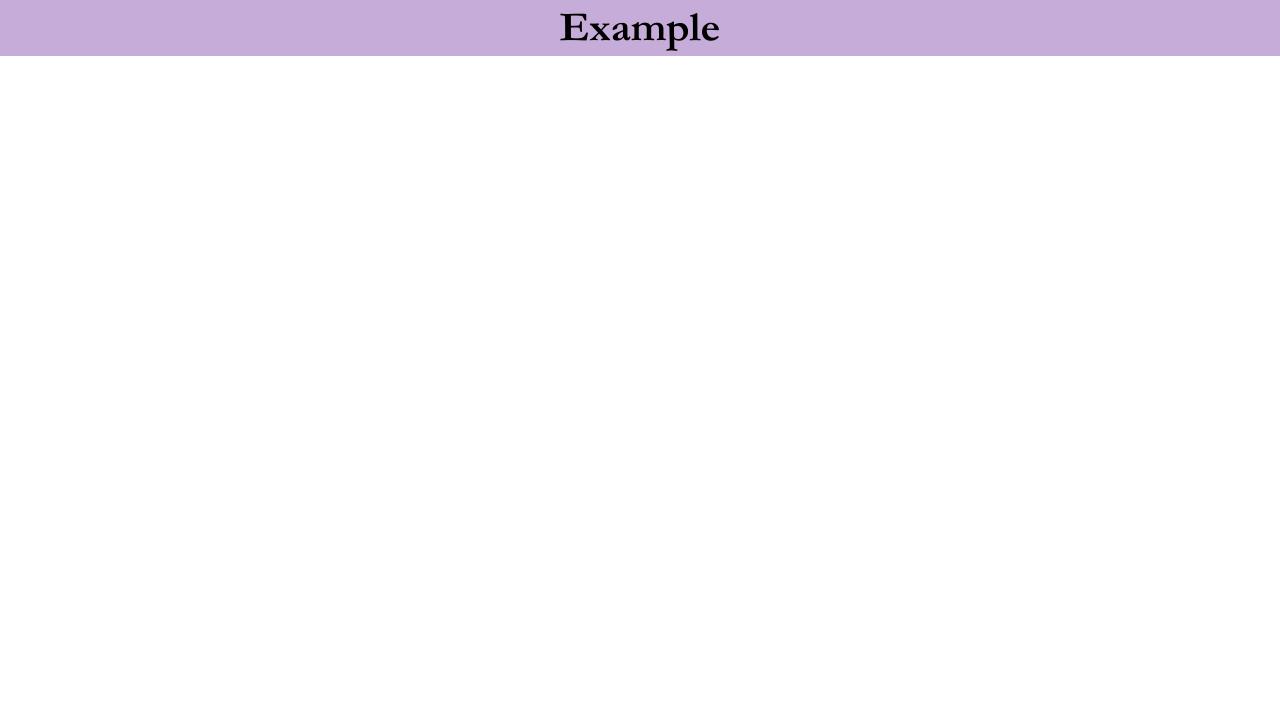
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

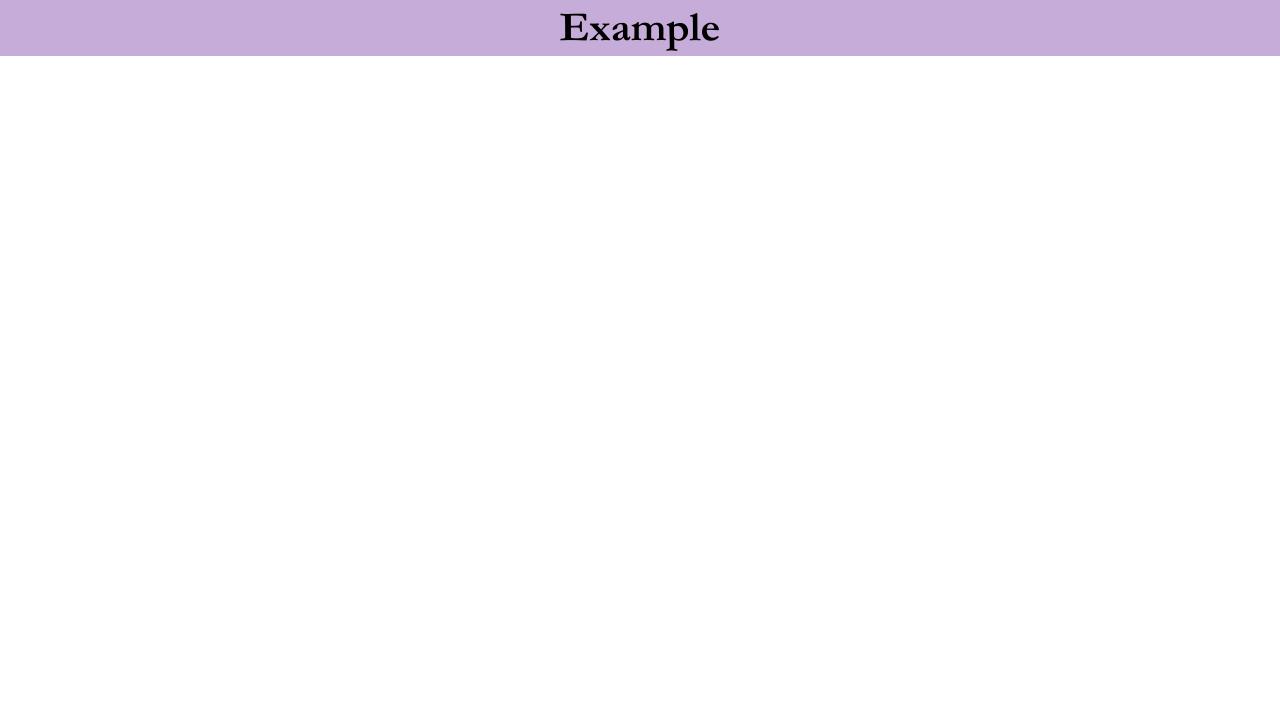
$$I \qquad \vec{x} = A^{-1} \qquad \vec{b}$$

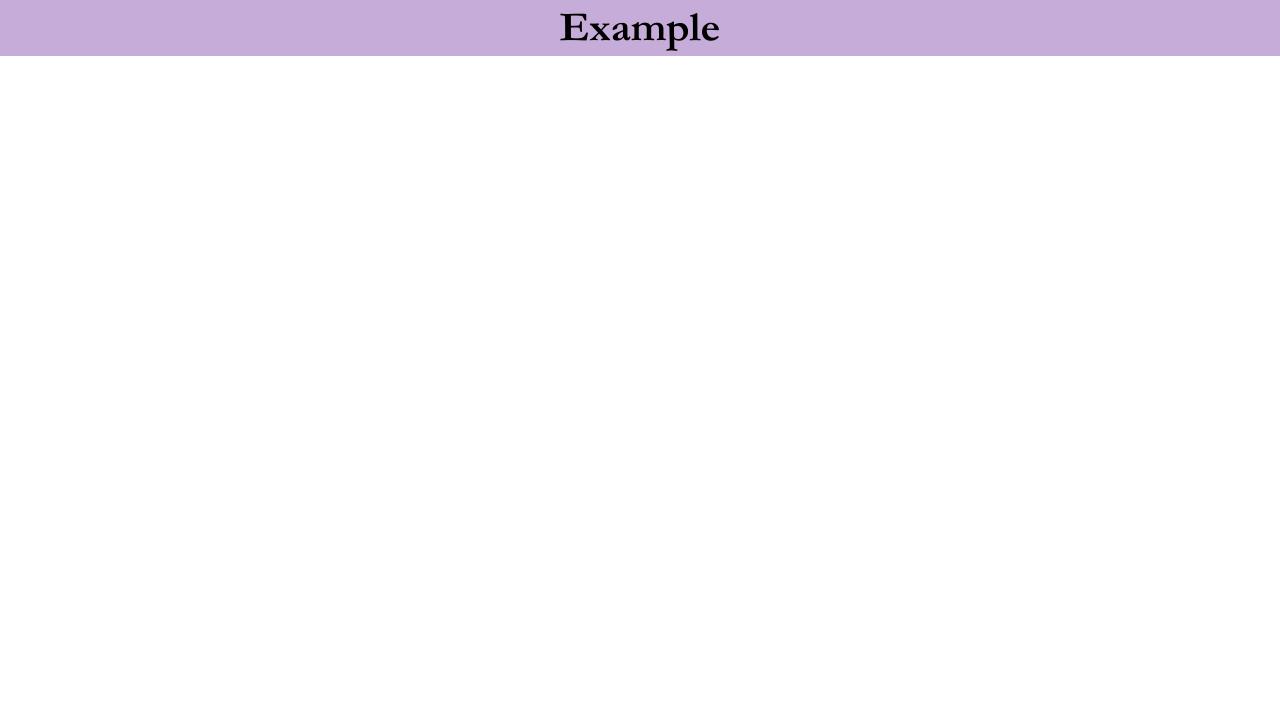


Example

Example: Find the inverse of
$$A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}$$
.









More on the Transpose of a Matrix

Recall

The transpose of an $m \times n$ matrix A is denoted by A^T , and it has entries $a_{ij}^T = a_{ji}$. That is, the columns of A^T are the rows of A.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \implies A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}$$

Properties of the Transpose

sum:
$$(A+B)^T = A^T + B^T$$

product:
$$(AB)^T = B^T A^T$$

inverse:
$$(A^T)^{-1} = (A^{-1})^T$$

Revisiting the Dot Product

We can redefine the dot product $\vec{u} \cdot \vec{v}$, where

$$ec{u} = \left[egin{array}{c} u_1 \ dots \ u_n \end{array}
ight], \; ec{v} = \left[egin{array}{c} v_1 \ dots \ v_n \end{array}
ight] \in \mathbb{R}^n,$$

as a matrix product $\vec{u}^T \vec{v}$.

$$\vec{u}^T \vec{v} = \underbrace{\begin{bmatrix} u_1 \ u_2 \ \dots \ u_n \end{bmatrix}}_{\mathbf{1} \times \mathbf{n}} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\mathbf{1} \times \mathbf{n}} \underbrace{= \underbrace{u_1 v_1 + u_2 v_2 + \dots + u_n v_n}_{\mathbf{1} \times \mathbf{1}}}_{\mathbf{1} \times \mathbf{1}}$$



Properties of Inverses

Inverse of a Product

Theorem: If A and B are invertible, then AB is invertible, with

$$(AB)^{-1} = B^{-1}A^{-1}$$

Inverse of the sum of Matrices

In general, even if both A and B are invertible matrices of the same size, the matrix (A + B) is not necessarily invertible.

Inverse of a Diagonal Matrix

Let
$$D = \left[egin{array}{ccc} d_{11} & & & & \\ & d_{22} & & & \\ & & \ddots & \\ & & d_{nn} \end{array}
ight]$$
 be an $n imes n$ diagonal matrix, then

$$D^{-1} = \begin{bmatrix} 1/d_{11} & & & \\ & 1/d_{22} & & \\ & & \ddots & \\ & & & 1/d_{nn} \end{bmatrix} \text{ provided that } d_{ii} \neq 0.$$

Inverse of an Elimination Matrix

Consider the elimination matrix

$$E_{31} = \left[egin{array}{ccccc} 1 & 0 & 0 & \dots & 0 \ 0 & 1 & 0 & \dots & 0 \ c & 0 & 1 & \dots & 0 \ dots & \ddots & & & \ 0 & 0 & 0 & \dots & 1 \end{array}
ight]$$

which adds c copies of the first row to the third row. Then,

$$E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ -c & 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

Inverse of a Permutation Matrix

The inverse of a permutation matrix is its transpose.

$$P_{34} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \implies P_{34}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \implies P^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Inverse of upper triangular matrix

Block multiplication If the cuts between columns of A match the cuts between rows of B, then block multiplication of AB is allowed:

$$\begin{bmatrix}
n_1 & n_2 \\
m_1 \begin{bmatrix} A_{11} & A_{12} \\
A_{21} & A_{22} \end{bmatrix} \begin{bmatrix}
B_{11} & \cdots \\
B_{21} & \cdots \end{bmatrix} \stackrel{n_1}{=} \begin{bmatrix}
n_{A_{11}} & k_1 & n_2 & k_1 \\
m_{A_{11}} & B_{11} & h_{11} & h_{12} & B_{21} & \cdots \\
m_{A_{21}} & B_{11} & h_{12} & B_{21} & \cdots \end{bmatrix} .$$
(1)



Questions?