NYU

Linear Algebra

Lecture 5

Inverse Matrices

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Recap

Block multiplication If the cuts between columns of A match the cuts between rows of B, then block multiplication of AB is allowed:

$$\begin{array}{c} \\ m_1 \\ m_2 \end{array}\begin{bmatrix} \overset{n_1}{A_{11}} & \overset{n_2}{A_{12}} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \overset{k_1}{B_{11}} & \cdots \\ B_{21} & \cdots \end{bmatrix} \begin{array}{c} n_1 \\ n_2 \end{array} = \begin{array}{c} m_1 \\ m_2 \end{array}\begin{bmatrix} \overset{n_1}{A_{11}}\overset{k_1}{B_{11}}{}^{n_1} + \overset{n_2}{A_{12}}\overset{k_1}{B_{21}}{}^{n_2} & \cdots \\ A_{21}B_{11} + A_{22}B_{21} & \cdots \end{bmatrix}. \qquad (1)$$

Strang Sections 2.5 – Inverse Matrices

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text

The Idea of Inverse Matrices

The idea of Inverse Matrices

Suppose A is an $n \times n$ matrix (square matrix), then A is invertible if there exists a matrix A^{-1} such that

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I.$$

We can only talk about an inverse of a square matrix, but not all square matrices are invertible. We will discuss such restrictions in future lectures.

The idea of Inverse Matrices

Recall: The multiplicative inverse (or reciprocal) of a nonzero number a is the number b such that $ab = 1$. We define the inverse of a matrix in almost the same way.

Definition

Let A be an $n \times n$ square matrix. We say A is **invertible** (or **nonsingular**) if there is a matrix B of the same size, such that

$$AB = I_n \quad \text{and} \quad BA = I_n.$$

identity matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

In this case, B is the **inverse** of A, and is written A^{-1}.

Example

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

I claim $B = A^{-1}$. Check:

The idea of Inverse Matrices

Consider the following system:

$$
\begin{aligned}
2x_1 + 4x_2 - 2x_3 &= 2 \\
4x_1 + 9x_2 - 3x_3 &= 8 \\
-2x_1 - 3x_2 + 7x_3 &= 10
\end{aligned}
$$

Our goal is to find x_1, x_2, and x_3. In matrix form, this system is:

$$
\underbrace{\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}
$$

$$
A \quad \vec{x} = \vec{b}
$$

idea →

$$
\vec{x} = A^{-1} \vec{b}
$$

The idea of Inverse Matrices

$$\underbrace{\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

$$A \quad \vec{x} = \vec{b}$$

$$\iff \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

$$A \quad \vec{x} = I \quad \vec{b}$$

idea →

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

$$I \quad \vec{x} = A^{-1} \quad \vec{b}$$

The idea of Inverse Matrices

$$\left[\begin{array}{c|c} A & I \end{array}\right]$$

$$\Downarrow \textbf{elimination}$$

$$\left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$$

Example

Example: Find the inverse of $A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}$.

Example

Example

Example

More on the Transpose of a Matrix

Recall

The transpose of an $m \times n$ matrix A is denoted by A^T, and it has entries $a^T_{ij} = a_{ji}$. That is, the columns of A^T are the rows of A.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \Longrightarrow A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & & & \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

Properties of the Transpose

sum: $(A+B)^T = A^T + B^T$

product: $(AB)^T = B^T A^T$

inverse: $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$

Revisiting the Dot Product

We can redefine the dot product $\vec{u} \cdot \vec{v}$, where

$$\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n,$$

as a matrix product $\vec{u}^T \vec{v}$.

$$\vec{u}^T \, \vec{v} = \underbrace{\begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix}}_{1 \times n} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{n \times 1} = \underbrace{u_1 v_1 + u_2 v_2 + \cdots + u_n v_n}_{1 \times 1}$$

Properties of Inverses

Inverse of a Product

Theorem: If A and B are invertible, then AB is invertible, with

$$(AB)^{-1} = B^{-1}A^{-1}$$

Inverse of the sum of Matrices

In general, even if both A and B are invertible matrices of the same size, the matrix $(A+B)$ is not necessarily invertible.

Inverse of a Diagonal Matrix

Let $D = \begin{bmatrix} d_{11} & & & \\ & d_{22} & & \\ & & \ddots & \\ & & & d_{nn} \end{bmatrix}$ be an $n \times n$ diagonal matrix, then

$$D^{-1} = \begin{bmatrix} 1/d_{11} & & & \\ & 1/d_{22} & & \\ & & \ddots & \\ & & & 1/d_{nn} \end{bmatrix}$$ provided that $d_{ii} \neq 0$.

Inverse of an Elimination Matrix

Consider the elimination matrix

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ c & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

which adds c copies of the first row to the third row. Then,

$$E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ -c & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

Inverse of a Permutation Matrix

The inverse of a permutation matrix is its transpose.

$$P_{34} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Longrightarrow P_{34}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow P^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Inverse of upper triangular matrix

Block multiplication If the cuts between columns of A match the cuts between rows of B, then block multiplication of AB is allowed:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & \cdots \\ B_{21} & \cdots \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & \cdots \\ A_{21}B_{11} + A_{22}B_{21} & \cdots \end{bmatrix}. \tag{1}$$

Questions?