Lecture 13: Strong Convergence Theorem and Derivative Free Schemes

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Abstract

Convergence theorem for general strong Taylor schemes. Derivative free schemes.

Recall estimate in Lecture 12,

$$R_{0,u} = E(\sup_{s\in[0,u]} |g(s)|^2 \Big| A_0) < \infty, \tag{0.1}$$

$$F_t^\alpha = E(\sup_{z\in[0,t]} | \sum_{n=0}^{n_z-1} I_\alpha[g(\cdot)]_{\tau_n,\tau_{n+1}} + I_\alpha[g(\cdot)]_{\tau_{n_z},z}|^2 \Big| A_0). \tag{0.2}$$

Then w.p. 1 for $t \in [0,T]$:

$$F_t^\alpha \le t\delta^{2(l(\alpha)-1)} \int_0^t R_{0,u}\, du, \ \text{ if } \ l(\alpha) = n(\alpha), \tag{0.3}$$

and

$$F_t^\alpha \le 4^{l(\alpha)-n(\alpha)+2} \delta^{l(\alpha)+n(\alpha)-1} \int_0^t R_{0,u}\, du, \ \ l(\alpha) \ne n(\alpha), \tag{0.4}$$

Taking,

$$p(\alpha) = \begin{cases} 2(l(\alpha)-1), \text{ if } l(\alpha) = n(\alpha) \\ l(\alpha) + n(\alpha) - 1, \text{ if } l(\alpha) \ne n(\alpha), \end{cases} \tag{0.5}$$

we have,

$$F_t^\alpha \le C(T,\alpha)\delta^{p(\alpha)} \int_0^t R_{0,u}\, du. \tag{0.6}$$

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1 General Strong Convergence Theorem

Let $\alpha = (\alpha_1, \alpha_2, \cdots) \neq v$, v the empty index, δ the time step of discretization over $[0, T]$, τ_n's the uniform discrete time steps, $Y(t) = Y^\delta(t)$ the strong Ito-Taylor approximation of order $\gamma \geq 0.5$ satisfying:

$$Y(t) = Y_{n_t} + \sum_{\alpha \in A_\gamma} I_\alpha[f_\alpha(\tau_{n_t}, Y_{n_t})]_{\tau_{n_t},t} = Y_{n_t} + \sum_{\alpha \in A_\gamma} f_\alpha(\tau_{n_t}, Y_{n_t}) I_\alpha[1]_{\tau_{n_t},t}. \tag{1.7}$$

where

$$Y_{n+1} = Y_n + \sum_{\alpha \in A_\gamma} I_\alpha[f_\alpha(\tau_n, Y_n)]_{\tau_n,\tau_{n+1}} = Y_n + \sum_{\alpha \in A_\gamma} f_\alpha(\tau_n, Y_n) I_\alpha[1]_{\tau_n,\tau_{n+1}}, \tag{1.8}$$

$$n_z := \max\{n \in N | \tau_n \leq z\}. \tag{1.9}$$

Recall:

$$A_\gamma = \{\alpha : l(\alpha) + n(\alpha) \leq 2\gamma, \text{ or } l(\alpha) = n(\alpha) = \gamma + \frac{1}{2}\}, \tag{1.10}$$

with remainder (boundary) set $B = B(A) = \{\alpha \notin A : -\alpha \in A\}$.

Assume that for all α (detailed assumption can be found on KL's book):

$$|f_\alpha(t, x) - f_\alpha(t, y)| \leq K_1|x - y|, \tag{1.11}$$

$$|f_\alpha(t, x)| \leq K_2(1 + |x|), \tag{1.12}$$

then:

$$E(\sup_{t \in [0,T]} |X_t - Y^\delta(t)|^2 | A_0) \leq K_3(1 + |X_0|^2)\delta^{2\gamma} + K_4|X_0 - Y^\delta(0)|^2. \tag{1.13}$$

The constants K_i's are independent of δ.

1.1 Proof

The exact Ito-Taylor expansion is:

$$X_\tau = X(\rho) + \sum_{\alpha \in A_\gamma} I_\alpha[f_\alpha(\rho, X_\rho)]_{\rho,\tau} + \sum_{\alpha \in B(A_\gamma)} I_\alpha[f_\alpha(\cdot, X_\cdot)]_{\rho,\tau}, \tag{1.14}$$

for any $\rho < \tau$. It follows that:

$$\begin{aligned} X_t &= X_0 + \sum_{\alpha \in A_\gamma} (\sum_{n=0}^{n_t - 1} I_\alpha[f_\alpha(\tau_n, X_{\tau_n})]_{\tau_n,\tau_{n+1}} + I_\alpha[f_\alpha(\tau_{n_t}, X_{\tau_{n_t}})]_{\tau_{n_t},t}) \\ &+ \sum_{\alpha \in B(A_\gamma)} (\sum_{n=0}^{n_t - 1} I_\alpha[f_\alpha(\cdot, X_\cdot)]_{\tau_n,\tau_{n+1}} + I_\alpha[f_\alpha(\cdot, X_\cdot)]_{\tau_{n_t},t}). \end{aligned} \tag{1.15}$$

Take the difference of (1.15) and $Y(t)$ generate by (1.7):

$$\begin{aligned} Z(t) &= E(\sup_{s\in[0,t]} |X_s - Y^\delta(s)|^2 | A_0) \\ &\le C(|X_0 - Y^\delta(0)|^2 + \sum_{\alpha\in A_\gamma} R_t^\alpha + \sum_{\alpha\in B(A_\gamma)} U_t^\alpha). \end{aligned} \tag{1.16}$$

The two type of terms R^α and U^α are bounded below. First by (0.6), then Lipschitz condition on f_α:

$$\begin{aligned} R_t^\alpha &= E(\sup_{s\in[0,t]} |\sum_{n=0}^{n_s-1} I_\alpha[f_\alpha(\tau_n, X_{\tau_n}) - f_\alpha(\tau_n, Y_n^\delta)]_{\tau_n,\tau_{n+1}} \\ &\quad + I_\alpha[f_\alpha(\tau_{n_s}, X_{\tau_{n_s}}) - f_\alpha(\tau_{n_s}, Y_{n_s}^\delta)]_{\tau_{n_s},s}|^2 | A_0) \\ &\le C \int_0^t E(\sup_{s\in[0,u]} |f_\alpha(\tau_{n_s}, X_{\tau_{n_s}}) - f_\alpha(\tau_{n_s}, Y_{n_s}^\delta)|^2 | A_0) du \\ &\le C K_1^2 \int_0^t Z(u)\, du. \end{aligned} \tag{1.17}$$

Note that $\alpha \in A_\gamma$ here, so $l(\alpha)$ starts from 1.

Next for $\alpha \in B(A_\gamma)$:

$$\begin{aligned} U_t^\alpha &= E(\sup_{s\in[0,t]} |\sum_{n=0}^{n_s-1} I_\alpha[f_\alpha(\cdot, X_\cdot)]_{\tau_n,\tau_{n+1}} + I_\alpha[f_\alpha(\cdot, X_\cdot)]_{\tau_{n_s},s}|^2 | A_0) \\ &\le C_1(1 + |X_0|^2)\delta^{p(\alpha)}. \end{aligned} \tag{1.18}$$

For $\alpha \in B(A_\gamma)$, recall, $A_\gamma = \{\alpha : l(\alpha) + n(\alpha) \le 2\gamma, \text{ or } l(\alpha) = n(\alpha) = \gamma + \frac{1}{2}\}$

- if $l = n$:
  - if γ is not integer, $l \ge \gamma + 1/2 + 1$,
  - if γ is integer, $l \ge \gamma + 1$;
- if $l \ne n$, $l + n \ge 2\gamma + 1$.

So $p(\alpha) \ge 2\gamma$.

$$U^\alpha \le C_2(1 + |X_0|^2)\delta^{2\gamma}. \tag{1.19}$$

Finally, Z obeys inequality:

$$Z(t) \le C_2|X_0 - Y^\delta(0)|^2 + C_3(1 + |X_0|^2)\delta^{2\gamma} + C_4 \int_0^t Z(u)\, du, \tag{1.20}$$

by Gronwall inequality:

$$Z(T) \le C_5(1 + |X_0|^2)\delta^{2\gamma} + C_6|X_0 - Y^\delta(0)|^2. \tag{1.21}$$

2 Remarks for General Strong Convergence

Now if we further assume,

$$E\left(|X_0|^2\right) < \infty, \tag{2.22}$$

and

$$\sqrt{E\left(\left|X_0 - Y^\delta(0)\right|^2\right)} \leq K_5\delta^\gamma. \tag{2.23}$$

Then,

$$\sqrt{E\left(\sup_{0\leq t\leq T}\left|X_t - Y^\delta(t)\right|^2\right)} \leq K_6\delta^\gamma. \tag{2.24}$$

L_1 convergence Recall the Lyapunov inequality: if X is not concentrated on a single point and if $E\left(|X|^s\right)$ exists for some $s > 0$, then for all $0 < r < s$ and $a \in \Re$

$$\left(E\left(|X-a|^r\right)\right)^{1/r} \leq \left(E\left(|X-a|^s\right)\right)^{1/s} \tag{2.25}$$

Also we should realize,

$$\sup_{0\leq t\leq T}\left|X_t - Y^\delta(t)\right|^2 = \left(\sup_{0\leq t\leq T}\left|X_t - Y^\delta(t)\right|\right)^2 \tag{2.26}$$

so

$$E\left(\sup_{0\leq t\leq T}\left|X_t - Y^\delta(t)\right|\right) \leq K_6\delta^\gamma \tag{2.27}$$

L_p convergence In fact (need some derivation), given,

$$E\left(|X_0|^p\right) < \infty, \quad \left(E\left(\left|X_0 - Y^\delta(0)\right|^p\right)\right)^{1/p} \leq K_5^*\delta^\gamma \tag{2.28}$$

we have

$$\left(E\left(\sup_{0\leq t\leq T}\left|X_t - Y^\delta(t)\right|^p\right)\right)^{1/p} \leq K_6^*\delta^\gamma \tag{2.29}$$

Selection of p **If** in addition, on each time interval $[\tau_n, \tau_{n+1}]$ for each $n = 0, 1, \ldots$ and all $\alpha \in \mathcal{A}_\gamma$,

$$E\left(\left|I_\alpha - I_\alpha^p\right|^2\right) \leq K_7\delta^{2\gamma+1}, \tag{2.30}$$

then (left as project V),

$$E\left(\left|X_T - Y^{\delta}(T)\right|\right) \leq K_8 \delta^{\gamma}. \quad (2.31)$$

Recall,

$$E\left(|I_\alpha - I_\alpha^p|^2\right) \leq C\frac{\delta^2}{p}, \quad (2.32)$$

to obtain a strong scheme of order $\gamma = 1.0, 1.5$ or 2.0 we need to choose p so that (2.30) holds, that is equivalent to,

$$p \geq p(\delta) = \frac{C}{K_7}\delta^{1-2\gamma}. \quad (2.33)$$

3 Derivative Free Schemes

3.1 Convergence

RK (Runge-Kutta) schemes are constructed by approximating derivatives of drift and diffusion functions in Ito-Taylor expansion. Denote such a scheme as:

$$Y_{n+1} = Y_n + \sum_{\alpha \in A_\gamma} g_{\alpha,n} I_\alpha + R_n, \quad (3.34)$$

$$A_\gamma = \{\alpha : l(\alpha) + n(\alpha) \leq 2\gamma,\ or\ l(\alpha) = n(\alpha) = \gamma + \frac{1}{2}\}. \quad (3.35)$$

Still

$$p(\alpha) = \begin{cases} 2(l(\alpha) - 1),\ \text{if } l(\alpha) = n(\alpha) \\ l(\alpha) + n(\alpha) - 1,\ \text{if } l(\alpha) \neq n(\alpha), \end{cases} \quad (3.36)$$

Assume that:

$$E(\max_{0 \leq n \leq n_T} |g_{\alpha,n} - f_\alpha(Y_n)|^2) \leq K\delta^{2\gamma - p(\alpha)}, \quad (3.37)$$

and

$$E(\max_{1 \leq n \leq n_T} |\sum_{0 \leq k \leq n-1} R_k|^2) \leq K\delta^{2\gamma}. \quad (3.38)$$

A scheme satisfying the above is called *strong Ito scheme of order* γ .

Proof Compare Ito scheme with Ito-Taylor approximation of the same order. Denote the latter by $\bar{Y}_n$:

$$\bar{Y}_{n+1} = \bar{Y}_n + \sum_{\alpha \in A_\gamma} f_\alpha(\bar{Y}_n) I_\alpha,$$

converging with order γ to SDE solution X_t.

$$\begin{aligned} H_t &= E(\max_{n\in[0,n_t]} |\bar{Y}_n - Y_n|^2) \\ &= E(\max_{n\in[0,n_t]} |\sum_{k=0}^{n-1}\sum_{\alpha} f_\alpha(\bar{Y}_k)I_\alpha - \sum_{k=0}^{n-1}(\sum_{\alpha} g_{\alpha,k}I_\alpha + R_k)|^2) \\ &\leq K_1\sum_{\alpha}[E(\max_{n\in[0,n_t]} |\sum_{k=0}^{n-1}(f_\alpha(Y_k) - f_\alpha(\bar{Y}_n))I_\alpha|^2) + E(\max_{n\in[0,n_t]} |\sum_{k=0}^{n-1}(f_\alpha(Y_k) - g_{\alpha,k})I_\alpha|^2)] \\ &\quad +K_1E(\max_{n\in[0,n_t]} |\sum_{k=0}^{n-1} R_k|^2), \end{aligned} \tag{3.39}$$

for $t \in [0, T]$.

By (0.6) and (3.38),

$$\begin{aligned} H_t &\leq K_2\sum_{\alpha}[\int_0^t E(\max_{n\in[0,n_u]} |f_\alpha(Y_n) - f_\alpha(\bar{Y}_n)|^2 du) \\ &\quad + \int_0^t E(\max_{n\in[0,n_u]} |f_\alpha(Y_n) - g_{\alpha,n}|^2 du)]\delta^{p(\alpha)} + K_2\delta^{2\gamma} \\ &\leq K_3\int_0^t H_u du + K_4\delta^{2\gamma}, \end{aligned} \tag{3.40}$$

where the second step comes from Lipschitz condition on f_α and (3.37). Finally by Gronwall inequality,

$$E\left(\max_{0\leq n\leq n_T} \left|Y_n - \bar{Y}_n\right|^2\right) \leq K\delta^{2\gamma}. \tag{3.41}$$

Note,

$$|Y_n - X_{\tau_n}|^2 \leq 2\left|Y_n - \bar{Y}_n\right|^2 + 2\left|\bar{Y}_n - X_{\tau_n}\right|^2, \tag{3.42}$$

so

$$E\left(\max_{0\leq n\leq n_T} |Y_n - X_{\tau_n}|^2\right) \leq K\delta^{2\gamma} \tag{3.43}$$

3.2 Platen scheme

Platen scheme is:

$$\begin{aligned} Y_{n+1} &= Y_n + a\Delta + b\Delta W_n + \frac{1}{2\sqrt{\Delta}}(b(Y_n^*) - b)((\Delta W_n)^2 - \Delta), \\ Y_n^* &= Y_n + a\Delta + b\sqrt{\Delta}. \end{aligned} \tag{3.44}$$

It can be written as:

$$Y_{n+1} = Y_n + g_{(0),n} I_{(0)} + g_{(1),n} I_1 + g_{(1,1),n} I_{(1,1)} + R_n, \tag{3.45}$$

$$\begin{aligned} R_n &= (b(Y_n^*) - b - bb'\sqrt{\Delta}) \times \frac{1}{2\sqrt{\Delta}}((\Delta W_n)^2 - \Delta) && (3.46) \\ &= [ab'\Delta + \frac{1}{2}b''(Y_n + \theta(a\Delta + b\sqrt{\Delta}))(a\Delta + b\sqrt{\Delta})^2] \times \frac{1}{2\sqrt{\Delta}}((\Delta W_n)^2 - \Delta), && (3.47) \end{aligned}$$

with:

$$g_{(0),n} = a = f_{(0)}, \; g_{(1),n} = b = f_{(1)}, g_{(1,1),n} = bb' = f_{(1,1)}.$$

Condition (3.37) holds exactly. To show (3.38), observe: $\sum_{k=0}^{n} R_k$ is mean zero, bounded variance, and is martingale. With Doob inequality:

$$\begin{aligned} E(\max_{n\in[0,n_T]} |\sum_{k=0}^{n-1} R_k|^2) &\leq K\Delta\, E(\max_{n\in[0,n_T]} |\sum_{k=0}^{n-1}((\Delta W_k)^2 - \Delta)|^2) \\ &\leq 4K\Delta \max_{n\in[0,n_T]} E(|\sum_{k=0}^{n-1}((\Delta W_k)^2 - \Delta)|^2) \\ &\leq 4K\Delta \sum_{k=0}^{n_T-1} E(|(\Delta W_k)^2 - \Delta|^2) \\ &= 8KT\Delta^2. && (3.48) \end{aligned}$$

The last equality we used $E(N(0,1))^4 = 3$. We conclude that Platen scheme is order one convergent.

3.3 Chang scheme

To simplify higher order schemes, consider $b =$ constant, in the so called additive noise regime. Then the 2nd order Chang scheme is:

$$Y_{n+1} = Y_n + \frac{1}{2}(a(\bar{Y}_+) + a(\bar{Y}_-))\Delta + b\Delta W_n. \tag{3.49}$$

$$\bar{Y}_\pm = Y_n + a\Delta/2 + \frac{1}{\Delta} b \cdot (\Delta Z \pm \sqrt{2J_{(1,1,0)}\Delta - (\Delta Z)^2}), \tag{3.50}$$

$$\Delta Z = \int_0^\Delta W_s ds, \quad J_{(1,1,0)} = \frac{1}{2}\int_0^\Delta W_s^2\, ds.$$

Cauchy-Schwartz:

$$2J_{(1,1,0)}\Delta - (\Delta Z)^2 \geq 0.$$

Note: $E((\Delta Z)^2) = \Delta^3/3$, $E(J_{(1,1,0)}) = \Delta^2/2$.

The Ito-Taylor expansion up to 2nd order is:

$$Y_{n+1} = Y_n + a\Delta + b\Delta W + ba'\Delta Z + aa'\Delta^2/2 + b^2 a'' J_{(1,1,0)}, \tag{3.51}$$

Taylor expansion of $a(\bar{Y}_\pm)$ about $a(Y_n)$ allows one to write (3.49) in the general RK form with $g_{\alpha,n} = f_{\alpha,n}$, $\alpha \in A_2$, and $E(R_n^2) = O(\Delta^5)$. Then (3.37) is true, and (3.38) holds with $\gamma = 2$. The Chang scheme is 2nd order accurate.

If $b = b(t)$, $a = a(t, x)$, the scheme extends to:

$$\begin{aligned} Y_{n+1} &= Y_n + \frac{1}{2}(a(t_n + \frac{1}{2}\Delta, \bar{Y}_+) + a(t_n + \frac{1}{2}\Delta, \bar{Y}_-))\Delta \\ &\quad + b(t_n)\Delta W + \frac{1}{\Delta}(b(t_{n+1}) - b(t_n))(\Delta W \Delta - \Delta Z). \end{aligned} \tag{3.52}$$

4 Project V: Due March 3 before lecture

V-1 Finish the proof in (2.31).

V-2 Consider, for $t \geq t_0 = 0$,

$$dX_t = \left(\frac{2}{1+t} X_t + (1+t)^2 \right) dt + (1+t)^2 dW_t \tag{4.53}$$

with initial value $X_0 = 1$.

(a) Verify it has exact solution,

$$X_t = (1+t)^2 \left(1 + W_t + t\right). \tag{4.54}$$

(b) Approximate X_T with scheme (3.52), for $T = 0.5$ in which $J_{(1,1,0)}$ is approximated by $J^p_{(1,1,0)}$ with $p = 15$. Conduct such approximation for equal step sizes $\delta = 2^{-1}, 2^{-2}, 2^{-3}, 2^{-4}, 2^{-5}$ and record the absolute error.

(c) Plot $\log_2$ of the absolute error against $\log_2 \delta$ and explain what you see.