Lecture 11: Strong Schemes with higher order

Zhongjian Wang*

Abstract

Constructing higher order strong schemes based on Ito-Taylor expansion and approximation of multiple stochastic integral.

Consider Ito Taylor expansion in the case $d = m = 1$ and the hierarchical set

$$\mathcal{A} = \{\alpha \in \mathcal{M} : l(\alpha) \leq 3\},$$

$$\begin{aligned}
X_t = X_{t_0} &+ aI_{(0)} + bI_{(1)} + \left(aa' + \frac{1}{2}b^2a''\right) I_{(0,0)} \\
&+ \left(ab' + \frac{1}{2}b^2b''\right) I_{(0,1)} + ba'I_{(1,0)} + bb'I_{(1,1)} \\
&+ \left[a\left(aa'' + (a')^2 + bb'a'' + \frac{1}{2}b^2a'''\right) + \frac{1}{2}b^2\left(aa''' + 3a'a''\right.\right. \\
&+ \left.\left.\left((b')^2 + bb''\right)a'' + 2bb'a'''\right) + \frac{1}{4}b^4a^{(4)}\right] I_{(0,0,0)} \\
&+ \left[a\left(a'b' + ab'' + bb'b'' + \frac{1}{2}b^2b'''\right) + \frac{1}{2}b^2\left(a''b' + 2a'b''\right.\right. \\
&\left.\left. + ab''' + \left((b')^2 + bb''\right)b'' + 2bb'b''' + \frac{1}{2}b^2b^{(4)}\right)\right] I_{(0,0,1)} \\
&+ \left[a\left(b'a' + ba''\right) + \frac{1}{2}b^2\left(b''a' + 2b'a'' + ba'''\right)\right] I_{(0,1,0)} \\
&+ \left[a\left((b')^2 + bb''\right) + \frac{1}{2}b^2\left(b''b' + 2bb'' + bb'''\right)\right] I_{(0,1,1)} \\
&+ b\left(aa'' + (a')^2 + bb'a'' + \frac{1}{2}b^2a'''\right) I_{(1,0,0)} \\
&+ b\left(ab'' + a'b' + bb'b'' + \frac{1}{2}b^2b'''\right) I_{(1,0,1)} \\
&+ b\left(a'b' + a''b\right) I_{(1,1,0)} + b\left((b')^2 + bb''\right) I_{(1,1,1)} + R. \qquad (0.1)
\end{aligned}$$

A rule of thumb for taking terms of (0.1) to build a scheme of certain accuracy is:

*Department of Statistics, University of Chicago

1. A truncated term I_α yields local truncation error $\mathcal{O}(\Delta)^{(n(\alpha)+l(\alpha))/2}$.
2. With local truncation error $\mathcal{O}(\Delta)^k$, a scheme yields global error $\mathcal{O}(\Delta)^{k-1}$.

Later on, we shall make rigorous this rule, by studying sums of stochastic integrals and their mean square estimates.

If one keeps the first three terms, one gets the Euler method accurate of order $O(t^{1/2})$.

1 Order 1 Milstein Scheme

By taking a noisy term $bb'I_{(1,1)}$, the accuracy goes up to order one. The Ito-Taylor expansion up to two layer integrals and order $O(t^{3/2})$ terms for autonomous diffusion process X_t:

$$\begin{aligned} X_t &= X_0 + aI_0 + bI_1 + (aa' + \frac{1}{2}b^2a'')I_{(0,0)} \\ &\quad +[ab' + \frac{1}{2}b^2b'']I_{(0,1)} + ba'I_{(1,0)} \\ &\quad +bb'I_{(1,1)} + b((b')^2 + bb'')I_{(1,1,1)} + \cdots , \end{aligned} \tag{1.2}$$

dots mean other higher layered integrals.

The Euler method is accurate of order $O(t^{1/2})$, and it is a truncation of (0.1) taking the first 3 terms. The Milstein method is constructed from (0.1) by taking an additional noisy term $bb'I_{(1,1)}$.

To evaluate the double integral $I_{(1,1)}$, recall the Ito to Stratonovich conversion formula:

$$\int_0^t h(W_s)dW_s = \int_0^t h(W_t) \circ dW_s - \frac{1}{2}\int_0^t h'(W_s)ds, \tag{1.3}$$

for any C^1 function h. Then:

$$\begin{aligned} I_{(1,1)} &= \int_0^t W_s dW_s \\ &= \int_0^t W_s \circ dW_s - \frac{1}{2}\int_0^t ds \\ &= \frac{1}{2}(W_t^2 - t). \end{aligned} \tag{1.4}$$

Milstein scheme:

$$Y_{n+1} = Y_n + a\Delta + b\Delta W + \frac{1}{2}bb'((\Delta W)^2 - \Delta), \tag{1.5}$$

Δ time step, ΔW Brownian increment from t_{n-1} to t_n. We shall show that the Milstein scheme is strongly convergent of order $\gamma = 1$ for $a \in C^1$, $b \in C^2$. It is *the stochastic extension of deterministic Euler preserving the order of accuracy.*

In the general multi-dimensional case with $d, m = 1, 2, \ldots$ the k th component of the Milstein scheme has the form,

$$Y_{n+1}^k = Y_n^k + a^k \Delta + \sum_{j=1}^{m} b^{k,j} \Delta W^j + \sum_{j_1, j_2=1}^{m} L^{j_1} b^{k,j_2} I_{(j_1,j_2)}. \tag{1.6}$$

In which, we know,

$$I_{(j_1,j_1)} = \frac{1}{2} \left\{ \left(\Delta W^{j_1} \right)^2 - \Delta \right\}, \tag{1.7}$$

and for $j_1 \neq j_2$

$$\begin{aligned} I_{(j_1,j_2)} = J_{(j_1,j_2)} = & \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_1} dW_{s_2}^{j_1} dW_{s_1}^{j_2} \\ = & \frac{1}{2} W_\Delta^{j_1} W_\Delta^{j_2} - \frac{1}{2} \left(a_{j_2,0} W_\Delta^{j_1} - a_{j_1,0} W_\Delta^{j_2} \right) + \pi \sum_{r=1}^{\infty} r \left(a_{j_1,r} b_{j_2,r} - b_{j_1,r} a_{j_2,r} \right). \end{aligned} \tag{1.8}$$

It can be approximated by,

$$J_{(j_1,j_2)}^p = \Delta \left(\frac{1}{2} \xi_{j_1} \xi_{j_2} + \sqrt{\rho_p} \left(\mu_{j_1,p} \xi_{j_2} - \mu_{j_2,p} \xi_{j_1} \right) \right) \tag{1.9}$$

$$+ \frac{\Delta}{2\pi} \sum_{r=1}^{p} \frac{1}{r} \left(\zeta_{j_1,r} \left(\sqrt{2} \xi_{j_2} + \eta_{j_2,r} \right) - \zeta_{j_2,r} \left(\sqrt{2} \xi_{j_1} + \eta_{j_1,r} \right) \right) \tag{1.10}$$

where

$$\rho_p = \frac{1}{12} - \frac{1}{2\pi^2} \sum_{r=1}^{p} \frac{1}{r^2} \tag{1.11}$$

and $\xi_j, \mu_{j,p}, \eta_{j,r}$ and $\zeta_{j,r}$ are independent $N(0;1)$ Gaussian random variables with

$$\xi_j = \frac{1}{\sqrt{\Delta}} \Delta W^j.$$

How to choose p?

In general, we shall examine the mean-square error between J_α^p and J_α. The most sensitive approximation is $J_{(j_1,j_2)}^p$ because the others are either identical to J_α or *their* *mean-square error can be estimated by a constant times Δ^γ for some* $\gamma \geq 3$. We have

$$
\begin{aligned}
& E\left(\left|J^p_{(j_1,j_2)} - J_{(j_1,j_2)}\right|^2\right) \\
& = \Delta^2 E\left(A^p_{j_1,j_2} - A_{j_1,j_2}\right)^2 \\
& = \Delta^2 E\left(\frac{\pi}{\Delta}\sum_{r=p+1}^{\infty} r\left(a_{j_1,r}b_{j_2,r} - b_{j_1,r}a_{j_2,r}\right)\right)^2 \\
& (\text{Given } j_1 \neq j_2) \\
& = \pi^2\left(\sum_{r=p+1}^{\infty} r^2 E\left(a_{j_1,r}b_{j_2,r} - b_{j_1,r}a_{j_2,r}\right)^2\right) \\
& (\text{As } E\left(a^2_{j,r}\right) = E\left(b^2_{j,r}\right) = \frac{\Delta}{2r^2\pi^2}) \\
& = \frac{\Delta^2}{2\pi^2}\sum_{r=p+1}^{\infty}\frac{1}{r^2} \\
& \leq \frac{\Delta^2}{2\pi^2}\int_p^\infty \frac{1}{u^2}du = \frac{\Delta^2}{2\pi^2 p}
\end{aligned}
$$

We know the truncation error in Ito-Taylor expansion is $\mathcal{O}(\Delta^{3/2})$, so we expect

$$E\left(\left|J^p_{(j_1,j_2)} - J_{(j_1,j_2)}\right|^2\right) = \mathcal{O}(\Delta^3) \tag{1.12}$$

which yields,

$$p = p(\Delta) \geq \frac{K}{\Delta}. \tag{1.13}$$

2 Order 1.5 strong scheme

For the method to be order 1.5, include noisy terms in Ito-Taylor expansion up to order $O(t^{3/2})$, and deterministic terms of order $O(t^2)$, to be precise,

$$
\begin{aligned}
X_t &= X_0 + aI_0 + bI_1 + (aa' + \frac{1}{2}b^2a'')I_{(0,0)} \\
&\quad +[ab' + \frac{1}{2}b^2b'']I_{(0,1)} + ba'I_{(1,0)} \\
&\quad +bb'I_{(1,1)} + b((b')^2 + bb'')I_{(1,1,1)} + \cdots,
\end{aligned} \tag{2.14}
$$

dots mean other higher layered integrals.

$$
\begin{aligned}
I_{(1,1,1)} &= \int_0^t dW_s \int_0^s dW_{s_2} \int_0^{s_2} dW_{s_1} \\
&= \int_0^t dW_s \, \frac{1}{2}(W_s^2 - s) \\
&= \frac{1}{6}W_t^3 - \frac{1}{2}\int_0^t W_s ds - \frac{1}{2}\int_0^t s dW_s, \qquad (2.15)
\end{aligned}
$$

by the Ito to Stratonovich conversion formula:

$$
\int_0^t h(W_s)dW_s = \int_0^t h(W_t) \circ dW_s - \frac{1}{2}\int_0^t h'(W_s)ds, \qquad (2.16)
$$

for any C^1 function h. The last two terms add up to $-\frac{1}{2}tW_t$, hence:

$$
I_{(1,1,1)} = \frac{1}{3!}(W_t^3 - 3tW_t). \qquad (2.17)
$$

Order 1.5 scheme is:

$$
\begin{aligned}
Y_{n+1} &= Y_n + a\Delta + b\Delta + \frac{1}{2}bb'((\Delta W)^2 - \Delta) \\
&\quad + a'b\Delta Z + \frac{1}{2}(aa' + b^2 a''/2)\Delta^2 \\
&\quad + (ab' + b^2 b''/2)(\Delta W \Delta - \Delta Z) \\
&\quad + \frac{1}{3!}b(bb'' + (b')^2)((\Delta W)^2 - 3\Delta)\Delta W, \qquad (2.18)
\end{aligned}
$$

$$
\Delta Z = I_{(1,0)} = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_2} dW_{s_1} ds_2, \qquad (2.19)
$$

with properties:
(1) $E(\Delta Z) = 0$,
(2) variance $E((\Delta Z)^2) = \Delta^3/3$, covariance $E(\Delta Z \Delta W) = \frac{\Delta^2}{2}$.
In fact ($t = \Delta$):

$$
\begin{aligned}
Var(\Delta Z) &= E((\int_0^t W_s \, ds)^2) = E(\int_0^t \int_0^t W_s W_s' ds ds') \\
&= \int_0^t \int_0^t \min(s, s') ds ds' \\
&= \int_0^t ds \, (\int_0^s + \int_s^t) \min(s, s') ds' \\
&= \int_0^t ds \, (s^2/2 + (t-s)s) \\
&= t^3/3 = \Delta^3/3.
\end{aligned}
$$

$$
\begin{aligned}
E(\Delta Z \Delta W) &= \lim E(\sum_{i=1}^{N} W(j\delta)\delta W(\Delta)) \\
&= \lim E(\sum_{i=1}^{N}(j\delta)\delta \\
&= \lim_{\delta\to 0, \delta N=\Delta} \delta^2 N(N+1)/2 = \Delta^2/2.
\end{aligned}
$$

The pair $(\Delta W, \Delta Z)$ can then be generated by a pair of unit Gaussian r.v (U_1, U_2) as:

$$
\begin{aligned}
\Delta W &= U_1\sqrt{\Delta}, \\
\Delta Z &= \frac{1}{2}\Delta^{3/2}(U_1 + U_2/\sqrt{3}),
\end{aligned} \tag{2.20}
$$

3 Order 2 Scheme

Including order $O(t^2)$ terms in Ito-Taylor expansion, one could derive 2nd order accurate schemes. For simplicity, it is better to write it through Stratonovich-Taylor expansion derived in the same way as Ito-Taylor except the drift coefficient a is modified to $\underline{a} = a - \frac{1}{2}bb'$. The second order scheme is:

$$
\begin{aligned}
Y_{n+1} &= Y_n + \underline{a}\Delta + b\Delta W \\
&+ \frac{1}{2}bb'(\Delta W)^2 + b\underline{a}'\Delta Z \\
&+ \frac{1}{2}\underline{a}\underline{a}'\Delta^2 + \underline{a}b'(\Delta W\Delta - \Delta Z) \\
&+ \frac{1}{3!}b(bb')'(\Delta W)^3 + \frac{1}{4!}b(b(bb')')'(\Delta W)^4 \\
&+ \underline{a}(bb')'J_{(0,1,1)} + b(\underline{a}b')'J_{(1,0,1)} \\
&+ b(b\underline{a}')'J_{(1,1,0)},
\end{aligned} \tag{3.21}
$$

the $J_{(0,1,1)}$ etc are defined same as $I_{(0,1,1)}$ only with the integration in the sense of Stratonovich. The ΔW and ΔZ are the same as in 1.5 scheme.

$$
\begin{aligned}
&\Delta W = J^p_{(1)} = \sqrt{\Delta}\zeta_1, \quad \Delta Z = J^p_{(1,0)} = \frac{1}{2}\Delta\left(\sqrt{\Delta}\zeta_1 + a_{1,0}\right) \\
&J^p_{(1,0,1)} = \frac{1}{3!}\Delta^2\zeta_1^2 - \frac{1}{4}\Delta a_{1,0}^2 + \frac{1}{\pi}\Delta^{3/2}\zeta_1 b_1 - \Delta^2 B^p_{1,1} \\
&J^p_{(0,1,1)} = \frac{1}{3!}\Delta^2\zeta_1^2 - \frac{1}{2\pi}\Delta^{3/2}\zeta_1 b_1 + \Delta^2 B^p_{1,1} - \frac{1}{4}\Delta^{3/2}a_{1,0}\zeta_1 + \Delta^2 C^p_{1,1} \\
&J^p_{(1,1,0)} = \frac{1}{3!}\Delta^2\zeta_1^2 + \frac{1}{4}\Delta a_{1,0}^2 - \frac{1}{2\pi}\Delta^{3/2}\zeta_1 b_1 + \frac{1}{4}\Delta^{3/2}a_{1,0}\zeta_1 - \Delta^2 C^p_{1,1}
\end{aligned}
$$

with

$$
\begin{aligned}
a_{1,0} &= -\frac{1}{\pi}\sqrt{2\Delta}\sum_{r=1}^{p}\frac{1}{r}\xi_{1,r} - 2\sqrt{\Delta\rho_p}\mu_{1,p}, \quad \rho_p = \frac{1}{12} - \frac{1}{2\pi^2}\sum_{r=1}^{p}\frac{1}{r^2} \\
b_1 &= \sqrt{\frac{\Delta}{2}}\sum_{r=1}^{p}\frac{1}{r^2}\eta_{1,r} + \sqrt{\Delta\alpha_p}\phi_{1,p}, \quad \alpha_p = \frac{\pi^2}{180} - \frac{1}{2\pi^2}\sum_{r=1}^{p}\frac{1}{r^4} \\
B_{1,1}^{p} &= \frac{1}{4\pi^2}\sum_{r=1}^{p}\frac{1}{r^2}\left(\xi_{1,r}^2 + \eta_{1,r}^2\right) \\
C_{1,1}^{p} &= -\frac{1}{2\pi^2}\sum_{\substack{r,i=1\\ r\neq l}}^{p}\frac{r}{r^2-l^2}\left(\frac{1}{l}\xi_{1,r}\xi_{1,l} - \frac{l}{r}\eta_{1,r}\eta_{1,l}\right)
\end{aligned}
$$

where $\zeta_1, \xi_{1,r}\eta_{1,r}, \mu_{1,p}$ and $\phi_{1,p}$ for $r = 1, \ldots, p$ and $p = 1, 2, \ldots$ denote independent standard Gaussian random variables.