NYU

Lecture 6

LU and LDU Factorizations

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Strang Sections 2.6 – Elimination = Factorization: $A = LU$ and 2.7 – Transposes and Permutations

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text

LU Factorization

Example – Factorize (LU) the matrix A

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix} = L \cdot U$$

ellimination

upper traingular form.

Step 1

$$E_{21} = \begin{bmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{bmatrix}$$ ← Replace R2 by R2 + (-2) . R1

$$E_{21}\,A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1_{=3+(-2)\times 1} & 3 \\ 4 & 6 & 8 \end{bmatrix} \leftarrow 5+(-2)\times 1$$

Step 2

$$E_{31} = \begin{bmatrix} 1 & & \\ & 1 & \\ -4 & & 1 \end{bmatrix}$$ ← Replace R3 by R3 + (-4) · R1

$$E_{31}\,(E_{21}\,A) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 4 \end{bmatrix}$$

Example – Factorize (LU) the matrix A

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix}$$

$$E_{31}(E_{21}A) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 4 \end{bmatrix}$$

Step 3)

$$E_{32} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -2 & 1 \end{bmatrix}$$

$$E_{32}\left[E_{31}E_{21}A\right] = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix}}_{\text{upper Traingular Form } U}$$

$4+(-2)\times 3$

$$\overbrace{E_{32}E_{31}E_{21}}^{\text{Lower Traingular}} A = U$$

$$A = (E_{32}E_{31}E_{21})^{-1} \cdot U$$

order?

$$= \underbrace{(E_{21}^{-1}E_{31}^{-1}E_{32}^{-1})}_{L} \cdot U$$

Example – Factorize (LU) the matrix A

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix}$$

$R_2 - 2R_1$
$R_3 - 4R_1$

Step 1. $E_{21} = \begin{bmatrix} 1 & & \\ \boxed{-2} & 1 & \\ & & 1 \end{bmatrix}$ $\qquad E_{21}A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 4 & 6 & 8 \end{bmatrix} \leftarrow R_2 - 2R_1$

Step 2 $E_{31} = \begin{bmatrix} 1 & & \\ & 1 & \\ \boxed{-4} & & 1 \end{bmatrix}$ $\qquad E_{31}(E_{21}A) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 4 \end{bmatrix} \leftarrow R_3 - 4R_1$

R_1'
R_2'
R_3'

Step 3. $E_{32} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -2 & 1 \end{bmatrix}$ $\qquad E_{32}(E_{31}E_{21}A) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix} \leftarrow R_3' - 2R_2'$

$\Uparrow$
U

$E_{32}E_{31}E_{21}A = U \qquad A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}\,U$

$L\ U$ decomp

Example – Factorize (LU) the matrix A

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix}$$

order!

$$A = \underbrace{(E_{21}^{-1} \; E_{31}^{-1} \; E_{32}^{-1})}_{L} \cdot U$$

$$L = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 4 & 2 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 3 \\ & & -2 \end{bmatrix}$$

diag = 1

May not be 1

Last Step Calculate L

review the inverse of elimination matrix

$$E_{21} = \begin{bmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{bmatrix} \quad E_{31} = \begin{bmatrix} 1 & & \\ & 1 & \\ -4 & & 1 \end{bmatrix} \quad E_{32} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -2 & 1 \end{bmatrix}$$

$$E_{21}^{-1} = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ & & 1 \end{bmatrix} \quad E_{31}^{-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ 4 & & 1 \end{bmatrix} \quad E_{32}^{-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ & 2 & 1 \end{bmatrix}$$

easy

$$L = \underbrace{E_{21}^{-1} \cdot E_{31}^{-1} \cdot E_{32}^{-1}}_{\text{good order}} = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 4 & 2 & 1 \end{bmatrix}$$

! just need to copy non-zero element!!

just can be used for LU decomposition!!!!!

Example – Factorize (LU) the matrix A

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 4 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 3 \\ & & -2 \end{bmatrix}$$

IS LU decomposition is Unique?? No!!

$$A = \underline{L_1 U_1} = \underline{L_2 U_2}$$

$$\Downarrow$$

$$\underline{L_2^{-1} L_1} U_1 U_1^{-1} = L_2^{-1} L_2 \underline{U_2 U_1^{-1}}$$

$$\underbrace{\boxed{\underset{L.T}{L_2^{-1}}\ \underset{L.T.}{L_1}}}_{L.T.} = \underbrace{\boxed{\underset{U.T.}{U_2}\ \underset{U.T.}{U_1^{-1}}}}_{U.T.} = \text{a diagnol matrix } D!!$$

$$L_2^{-1} L_1 = U_2 U_1^{-1} = D$$

$$\Rightarrow L_1 = L_2 \cdot D$$

$$U_2 = D \cdot U_1$$

" LDU" decomposition

$$A = L \quad D \quad U$$

$$\begin{pmatrix} 1 & & \\ * & 1 & \\ * & \ddots & * & 1 \end{pmatrix} \quad \text{diag Matrix} \quad \begin{pmatrix} 1 & * & & * \\ & 1 & \ddots & * \\ & & & 1 \end{pmatrix}$$

LDU Factorization is Unique!!!!

Why is this Important?

Why are LU Factorizations Important?

Consider the system $A\boldsymbol{x} = \boldsymbol{b}$ with LU factorization $A = LU$. Then we have

$$\boxed{L \underbrace{U\boldsymbol{x}}_{=\boldsymbol{y}} = \boldsymbol{b}.}$$

$L \underbrace{Ux}_{y} = b$

easy to solve

Step1 — $L \cdot y = b$

Step2 — $U \cdot x = y$

easy to solve

Therefore we can perform (a now familiar) 2-step solution procedure:

1. Solve the lower triangular system $L\boldsymbol{y} = \boldsymbol{b}$ for $\boldsymbol{y}$ by forward substitution.

2. Solve the upper triangular system $U\boldsymbol{x} = \boldsymbol{y}$ for $\boldsymbol{x}$ by back substitution.

Moreover, consider the problem $AX = B$ (i.e., many different right-hand sides that are associated with the same system matrix). In this case we need to compute the factorization $A = LU$ only once, and then

$$AX = B \quad \Longleftrightarrow \quad LUX = B,$$

$Ly = b_1$

$$\begin{cases} l_{11} y_1 = b_1 \\ l_{21} y_1 + l_{22} y_2 = b_2 \\ l_{31} y_1 + l_{32} y_3 + l_{33} y_3 = b_3 \end{cases}$$

and we proceed as before:

1. Solve $LY = B$ by many forward substitutions (in parallel).

2. Solve $UX = Y$ by many back substitutions (in parallel).

Solving Systems of Equations

$$\begin{aligned} 7x_1 - \ \ 2x_2 + \ \ x_3 &= 12 \\ 14x_1 - \ \ 7x_2 - \ \ 3x_3 &= 17 \\ -7x_1 + 11x_2 + 18x_3 &= 5 \end{aligned}$$

Do it at recitation

Solving Systems of Equations

Solving Systems of Equations

LDU Factorization

Goal

$$L = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 4 & 2 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 3 \\ & & -2 \end{bmatrix}$$

diag = 1

May not be 1

$A = L \cdot U.$

$A = L \; D \cdot U$

diag

change U's diag to 1

$$D = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

$A = L \cdot D \cdot U$

(?)

$$\begin{array}{c|l} \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 3 \\ & & -1 \end{bmatrix} & \begin{array}{l} \text{row1}/1 \\ \text{row2}/1 \\ \text{row 3}/-2. \end{array} \end{array}$$

$$D = \begin{pmatrix} d_{11} & & \\ & d_{22} & \\ & & d_{33} \end{pmatrix}$$

$$D \cdot U = \begin{pmatrix} d_{11} \cdot U' \text{row 1} \\ d_{22} \cdot U' \text{row 2} \\ d_{33} \cdot U' \text{row 3} \end{pmatrix}$$

Let’s try this with an Example

Find the LDU factorization of $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

→ Find LU decomposition

→ Rewrite $U = D \cdot U'$

U' diag is 1

Let's try this with an Example – Find U

Find the LDU factorization of $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

LDU Decomposition

$$E_{21} = \begin{bmatrix} 1 & & \\ -1/2 & 1 & \\ & & 1 \end{bmatrix} \qquad E_{21}A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$2 - 1/2 \times 1$ $\quad 1 - 1/2 \times$

$$E_{32} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -2/3 & 1 \end{bmatrix} \qquad E_{32}(E_{21}A) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 0 & 4/3 \end{bmatrix}$$

$2 - 2/3 \times 1$

This is U

$$A = L \cdot U$$

$$L = \begin{bmatrix} 1 & & \\ 1/2 & 1 & \\ & 2/3 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3/2 & 1 \\ & & 4/3 \end{bmatrix} \begin{array}{l} \text{Row1}/2 \\ \text{Row2}/3/2 \\ \text{Row3}/4/3 \end{array} \rightarrow$$

$$D = \begin{bmatrix} 2 & & \\ & 3/2 & \\ & & 4/3 \end{bmatrix}$$

$$U' = \begin{bmatrix} 1 & 1/2 & 0 \\ & 1 & 2/3 \\ & & 1 \end{bmatrix}$$

Let’s try this with an Example – Find L

Find the LDU factorization of $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

Symmetric Matrices

What is a Symmetric Matrix?

If $\underset{\mathbb{R}^{n\times m}}{A} = \underset{\mathbb{R}^{m\times n}}{A^T}$. then it's a symmetric Matrix

A must be a square Matrix

ex. M. symmetric

$$M = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 0 & 5 & 1 \\ 3 & 5 & -7 & -1 \\ 0 & 1 & -1 & 4 \end{bmatrix}$$

M^T: →

Properties of a Symmetric Matrix

- The inverse of a symmetric matrix is symmetric.

If $A = A^T$. then $\boxed{(A^{-1})^T} = A^{-1}$

<u>pf</u>. need to check $(A^{-1})^T A = I$ $A(A^{-1})^T = I$

$(A^{-1})^T = (\underline{A^T})^{-1} = A^{-1}$

Properties of a Symmetric Matrix

- AA^T and A^TA are symmetric.

A can by any $\mathbb{R}^{n\times m}$

A A^T $\rightarrow \mathbb{R}^{n\times n}$ square matrix

$\mathbb{R}^{n\times m}$ $\mathbb{R}^{m\times n}$

$A^T \; A \rightarrow \mathbb{R}^{m\times m}$

$\mathbb{R}^{m\times n}$ $\mathbb{R}^{n\times m}$

need to check

$$(AA^T)^T = AA^T$$

$$= (A^T)^T \cdot A^T$$ by $(A\cdot B)^T = B^T \cdot A^T$

$(A^T)^T = A$

$$= A \cdot A^T$$

Properties of a Symmetric Matrix

- If A is symmetric, then elimination leads to $A = LDL^T$.

LDU

$A = L \cdot D \cdot U$

$\parallel$

$A^T = U^T \cdot D^T \cdot L^T$ ← another LDU Decomposition.

Lower Traingular   Upper Traingular

Because LDU Decomposition is Unique!

So. $L = U^T$. $U = L^T$. → $A = L \cdot D \cdot L^T$.

Let's try this with an Example – Find D

Find the LDU factorization of $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

is symmetric

$L = \begin{bmatrix} 1 & & \\ 1/2 & 1 & \\ & 2/3 & 1 \end{bmatrix}$

$D = \begin{bmatrix} 2 & & \\ & 3/2 & \\ & & 4/3 \end{bmatrix}$

$U' = \begin{bmatrix} 1 & 1/2 & 0 \\ & 1 & 2/3 \\ & & 1 \end{bmatrix}$

!! $U' = L^T$

$A = L \cdot D \cdot U'$

$= \underline{L \cdot D \cdot L^T}$ (!)

LDL decomposition!!!

Example

Factor the following symmetric matrices into $A = LDL^T$:

- $A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$ $\quad E_{21} = \begin{bmatrix} 1 & \\ -3 & 1 \end{bmatrix} \xrightarrow{}$ $\quad L = E_{21}^{-1} = \begin{bmatrix} 1 & \\ 3 & 1 \end{bmatrix}$

R2 - 3R1

$$E_{21} \cdot A = \begin{bmatrix} 1 & 3 \\ 0 & -7 \end{bmatrix} = U$$

$$\Downarrow$$

$$D = \begin{bmatrix} 1 & \\ & -7 \end{bmatrix} \qquad U' = \begin{bmatrix} 1 & 3 \\ & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & -7 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ & 1 \end{bmatrix}$$

Example

Factor the following symmetric matrices into $A = LDL^T$:

- $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

Example

Factor the following symmetric matrices into $A = LDL^T$:

- $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$PA = LU$

PA = LU?

So far, our assumption has been that we always have nonzero pivots. In case a pivot is zero, we can't use it to eliminate elements below it, so we have to exchange rows to find a nonzero pivot before we can start eliminating.

In this case, we can't find $A = LU$. We can, however, find $PA = LU$, where P corresponds to row exchanges done on A (in advance).

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}.$$

Find PA = LU for A below

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}.$$