Singular Value Decomposition. $A \in \mathbb{R}^{m\times n}$

- A^TA and AA^T are symmetric

$$\underset{m\times n}{A} = \underset{m\times m}{U}\ \underset{m\times n}{\Sigma}\ \underset{n\times n}{V^T}$$

orthogonal matrix — diag matrix — orthogonal matrix

$U^TU = I$. U is orthogonal.

- $A^TA = (U\Sigma V^T)^T(U\Sigma V^T) = V\Sigma^T U^TU\Sigma V^T = V\Sigma^T\Sigma V^T$

orthogonal — diag — orthogonal

- $AA^T = U\Sigma\Sigma^TU^T$

$U = [\vec{u}_1 \cdots \vec{u}_n]$ eigenvectors of AA^T.

diagonalization of A^TA

$V = [\vec{v}_1, \cdots \vec{v}_n]$ is the eigenvectors of A^TA

A is $\mathbb{R}^{2\times 3}$ $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ $\qquad \Sigma$ is $\mathbb{R}^{2\times 3}$ $\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \end{bmatrix}$

$\underset{2\times 3}{\Sigma}\ \underset{3\times 2}{\Sigma^T} \to 2\times 2$ is similar to AA^T (same eigenvalue) $= \begin{bmatrix} \lambda_1^2 & \\ & \lambda_2^2 \end{bmatrix}$ $\quad \lambda_1^2 \quad \lambda_2^2$

$\underset{3\times 2}{\Sigma^T}\ \underset{2\times 3}{\Sigma} \to 3\times 3$ is similar to A^TA (same eigenvalue) $\begin{bmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & 0 \end{bmatrix}$ $\quad \lambda_1^2, \lambda_2^2, 0$

2 : same $\quad$ 3-2 eigen is zero

AA^T: 2 eigen λ_1^2, λ_2^2

A^TA: 3 eigen $\lambda_1^2, \lambda_2^2, 0$

---

singular value!

Thm. $\text{rank}(A) = \text{rank}(AA^T) = \text{rank}(A^TA)$ = number of nonzeros in $\lambda_1 \cdots \lambda_n$

diagonal elements of Σ

Reminder $\text{rank}(A) = \text{rank}(A \cdot \text{invertible matrix})$

$A = U\Sigma V^T$

invertible $\Rightarrow \text{rank}(A) = \text{rank}(\Sigma)$ = # non-zero in singular value.

Fact. $\text{Nul}(AA^T) = \text{Nul}(A^T)$

$AA^Tx = 0 \iff A^Tx = 0$

① $A^Tx = 0 \Rightarrow AA^Tx = 0$. $\qquad AA^Tx = A(A^Tx) = A\vec{0} = \vec{0}$

② $AA^Tx = 0 \Rightarrow A^Tx = 0$.

"square function"

$AA^Tx = 0 \Rightarrow \underbrace{x^T \underset{\text{symmetric}}{AA^T} x}_{(A^Tx)^T} = 0 \Rightarrow (A^Tx)^T(A^Tx) = 0 \Rightarrow A^Tx = 0$

quadratic function of x.

$A^T = (1 . 2)$ $\quad (A^T x)^T (A^T x) = (x_1 + 2x_2)^2$

$A^T x = x_1 + 2x_2$

$A^T = \begin{pmatrix} 1. & 2 \\ 3. & 4 \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad A^T x = \begin{pmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{pmatrix}$

$(A^T x)^T (A^T x) = (x_1 + 2x_2 \,.\, 3x_1 + 4x_2) \begin{pmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{pmatrix}$

$= (x_1 + 2x_2)^2 + (3x_1 + 4x_2)^2.$

$$\begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix}$$

$Nul(AA^T) = Nul(A)$

SVD $\quad \underset{m\times n}{A} = \underset{m\times m}{U}\,\underset{m\times n}{\Sigma}\,\underset{n\times n}{V^T}$

non-zero singular value = rank

$$= \underbrace{\begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \end{bmatrix}}_{\text{orthogonal matrix}} \begin{bmatrix} \underbrace{\begin{matrix} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_r & \\ & & & 0 \\ & & & & \ddots \\ & & & & & 0 \end{matrix}}_{m} & \underbrace{\begin{matrix} 0 & \cdots & 0 \\ 0 & & 0 \\ 0 & & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{matrix}}_{n-m} \end{bmatrix} \begin{bmatrix} \vec{v}_1^T \\ \vdots \\ \vec{v}_n^T \end{bmatrix}$$

$\vec{u}_1 \cdots \vec{u}_m$ m orthonormal basis of $\mathbb{R}^m$

$\vec{v}_1 \cdots \vec{v}_n$ is n orthonormal basis of $\mathbb{R}^n$

- Similar to symmetric matrix

$$A = \lambda_1 \vec{u}_1 \vec{v}_1^T + \lambda_2 \vec{u}_2 \vec{v}_2^T + \cdots + \lambda_r \vec{u}_r \vec{v}_r^T \underbrace{+ 0\, \vec{u}_{r+1} \vec{v}_{r+1}^T + \cdots + 0\, \vec{u}_m \vec{v}_m^T}_{\text{all } 0}$$

what is $\vec{u}\vec{v}^T$ $\quad \vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$ $\quad \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ $\quad \vec{u} \in \mathbb{R}^m, \ \vec{v} \in \mathbb{R}^n$

$$\vec{u}\vec{v}^T = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = \begin{array}{c} \begin{bmatrix} u_1 v_1 & u_1 v_2 & \cdots & u_1 v_n \\ u_2 v_1 & u_2 v_2 & \cdots & u_2 v_n \\ \vdots & \vdots & & \vdots \\ u_m v_1 & u_m v_2 & \cdots & u_m v_n \end{bmatrix} \\ \begin{matrix} v_1 \vec{u} & \quad v_2 \vec{u} & \quad \cdots & \quad v_m \vec{u} \end{matrix} \end{array} \begin{matrix} u_1 \vec{v}^T \\ u_2 \vec{v}^T \\ \\ u_m \vec{v}^T \end{matrix}$$

rank 1 matrix

$\vec{u}\vec{v}^T$. $\quad Col(\vec{u}\vec{v}^T) = span\{\vec{u}\}$ $\quad row(\vec{u}\vec{v}^T) = span\{\vec{v}\}$

$$A = \lambda_1 \underbrace{\vec{u}_1 \vec{v}_1^T}_{\substack{m\times n \text{ matrix} \\ \text{rank } 1}} + \lambda_2 \underbrace{\vec{u}_2 \vec{v}_2^T}_{\substack{m\times n \text{ matrix} \\ \text{rank } 1}} + \cdots + \lambda_r \underbrace{\vec{u}_r \vec{v}_r^T}_{\substack{m\times n \text{ matrix} \\ \text{rank } 1}}$$

Example. Case $r = 2$

$$A = \lambda_1 \vec{u}_1 \vec{v}_1^T + \lambda_2 \vec{u}_2 \vec{v}_2^T \qquad \vec{u}_1 = \begin{bmatrix} u_{11} \\ \vdots \\ u_{1m} \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} u_{21} \\ \vdots \\ u_{2m} \end{bmatrix}$$

$$= \lambda_1 \begin{bmatrix} u_{11} \vec{v}_1^T \\ \vdots \\ u_{1m} \vec{v}_1^T \end{bmatrix} + \lambda_2 \begin{bmatrix} u_{21} \vec{v}_2^T \\ \vdots \\ u_{2m} \vec{v}_2^T \end{bmatrix} = \begin{bmatrix} \lambda_1 u_{11} \vec{v}_1^T + \lambda_2 u_{21} \vec{v}_2^T \\ \vdots \\ \lambda_1 u_{1m} \vec{v}_1^T + \lambda_2 u_{2m} \vec{v}_2^T \end{bmatrix}$$

$Row(A) = span\{v_1, v_2\}$ $\qquad$ all row vectors are linear combination of $\vec{v}_1 \vec{v}_2$

Thm.

- $v_1, v_2, \cdots v_r$ is the orthonormal basis of $Row(A)$
- $u_1, u_2, \cdots, u_r$ is the orthonormal basis of $Col(A)$
- $v_{r+1}, v_{r+2}, \cdots v_n$ is the orthonormal basis of $Nul(A)$
- $u_{r+1}, u_{r+2}, \cdots u_m$ is the orthonormal basis of left $Nul(A)$ $(Nul(A^T))$

$Row(A) \perp Nul(A)$

---

by using the fact $Nul(AA^T) = Nul(A^T)$ left Nul

$Col(AA^T)$

AA^T. eigenvalue λ_1^2 eigenvectors v_1

λ_2^2 v_2

$\vdots$

λ_r^2 v_r

0 v_{r+1}

$\vdots$

0 v_n

$AA^T = V\Sigma\Sigma^T V^T$

$diag(\lambda_1^2 \cdots \lambda_r^2\, 0 \cdots 0)$

$v_1 \cdots v_r$ forms

$AA^Tv_1 = \lambda_1^2 v_1$

$AA^Tv_2 = \lambda_2^2 v_2$

$\vdots$

$AA^Tv_r = \lambda_r^2 v_r$

$AA^Tv_{r+1} = 0$

$\vdots$

$AA^Tv_n = 0$

$v_{r+1} \cdots v_n$ forms $Nul(AA^T)$

How to compute SVD

Example, $A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$ → (singular value)^2

- $A^TA = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix} \rightarrow$ $\lambda_1 = 18$ $(1/\sqrt{2}, \ -1/\sqrt{2})$

  $\lambda_2 = 32$ $(1/\sqrt{2}, \ 1/\sqrt{2})$

$A = U \Sigma V^T$ $\quad \Sigma = \begin{pmatrix} \sqrt{32} & \\ & \sqrt{18} \end{pmatrix}$ $\quad V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

How to compute U $\quad \Sigma = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \end{bmatrix}$

$\vec{u}_1 = \frac{1}{\sigma_1} A \vec{v}_1 \qquad \vec{u}_2 = \frac{1}{\sigma_2} A \vec{v}_2$

$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T.$

$A\vec{v}_1 = \sigma_1 \vec{u}_1 \underbrace{\vec{v}_1^T \vec{v}_1}_{\vec{v}_1^T\vec{v}_1 = 1} + \underbrace{\sigma_2 \vec{u}_2 \underbrace{\vec{v}_2^T \vec{v}_1}_{\vec{v}_2^T\vec{v}_1 = 0} + \cdots + \sigma_r \vec{u}_r \underbrace{\vec{v}_r^T \vec{v}_1}_{\vec{v}_r^T\vec{v}_1 = 0}}_{\text{all terms here is zero!}}$ ⇐ orthonormal

fix any σ_i to be positive

$\Rightarrow A\vec{v}_1 = \sigma_1 \vec{u}_1$

$\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \frac{1}{\sqrt{32}} \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{u}_2 = \frac{1}{\sigma_2} A\vec{v}_2 = \ \cdots \ = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

- $A = U \Sigma V^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{32} & \\ & \sqrt{18} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

I don't know the sign

What if you compute U from A^TA's eigenvector after you set V?

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -\sqrt{32} & \\ & \sqrt{18} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

all possible answers

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{32} & \\ & \sqrt{18} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\sqrt{32} & \\ & -\sqrt{18} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

Second Question. If I my compute my U first, What is V

$\vec{v}_1 = \frac{1}{\sigma_1} A^T u_1$, $\vec{v}_2 = \frac{1}{\sigma_2} A^T u_2$, ~ . .

Try to prove by yourself !!!!