Singular Value Decomposition. $A \leftarrow$ all matrix $\mathbb{R}^{m\times n}$

Idea. A^TA and AA^T are always square and symmetric

SVD: $A = \underset{\text{orthogonal}}{\overset{m\times n}{U}}\ \underset{\text{"diag" matrix}}{\overset{m\times n}{\Sigma}}\ \underset{\text{orthogonal}}{\overset{n\times n}{V^T}}$

$- A^TA = (U\Sigma V^T)^T U\Sigma V^T = V\Sigma^T U^T U \Sigma V^T = V \underbrace{\Sigma^T\Sigma}_{\text{eigenvalues}} V^T$

V is eigenvectors of A^TA

have the same eigenvalue

$- AA^T = U \underbrace{\Sigma\Sigma^T}_{\text{eigenvalues}} U^T$

A^TA and $\Sigma^T\Sigma$ are similar, AA^T and $\Sigma\Sigma^T$ are also.

U is the eigenvectors of AA^T

$\Sigma = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \end{bmatrix} \rightarrow \Sigma\Sigma^T = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix}^{2\times 2}$ eigen: λ_1^2, λ_2^2 2 eigen

$\Sigma^T\Sigma = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{3\times 3}$ eigen $\lambda_1^2, \lambda_2^2, 0$ 3 eigen

These 2 are the same

(3-2) eigen is 0

$\Rightarrow$ 1) A^TA 3x3 matrix $\lambda_1^2, \lambda_2^2, 0$

AA^T 2x2 matrix $\lambda_1^2\ \lambda_2^2$

Thm. $Nul(AA^T) = Nul(A^T)$

if $AA^T\vec{x} = \vec{0} \iff A^T\vec{x} = \vec{0}$ (equivalence)

① "$\Leftarrow$" easier $A^Tx = \vec{0} \Rightarrow AA^Tx = A(A^Tx) = A\cdot\vec{0} = \vec{0}$

② "$\Rightarrow$" hard. $AA^T\vec{x} = \vec{0} \Rightarrow \underbrace{\vec{x}^T \underbrace{AA^T}_{\text{symmetry}} \vec{x}}_{\text{quadratic function of } \vec{x}} = 0 \Rightarrow \underbrace{(A^Tx)^T A^Tx}_{\text{square function!}} = 0$

$\Rightarrow \|A^T\vec{x}\| = 0 \Rightarrow A^T\vec{x} = \vec{0}$

- $\mathrm{Nul}(AA^T) = \mathrm{Nul}(A^T)$

- $\mathrm{rank}(AA^T) = \mathrm{rank}(A^TA) = \mathrm{rank}(A) = \mathrm{rank}(A^T)$

= number of non-zeros in the diag of Σ.
$(\lambda_1, \lambda_2, \cdots)$

Four Fundamental Subspace. (orthogonal basis) via SVD.

$$\underset{m\times n}{A} = \underset{m\times m}{U}\,\underset{m\times n}{\Sigma}\,\underset{n\times n}{V^T} = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \end{bmatrix} \left[\begin{array}{ccccc|ccc} \lambda_1 & & & & & 0 & \cdots & 0 \\ & \lambda_2 & & & & 0 & \cdots & 0 \\ & & \ddots & & & \vdots & & \vdots \\ & & & \lambda_r & & 0 & \cdots & 0 \\ & & & & 0 & 0 & \cdots & 0 \\ & & & & \ddots & \vdots & & \vdots \\ & & & & 0 & 0 & \cdots & 0 \end{array}\right] \begin{bmatrix} \vec{v}_1^T \\ \vdots \\ \vec{v}_n^T \end{bmatrix}$$

m $\quad$ $n-m$

$\vec{u}_1, \vec{u}_2 \cdots \vec{u}_m$ are orthonormal basis of $\mathbb{R}^m$

$\vec{v}_1 \cdots \vec{v}_n$ are orthonormal basis of $\mathbb{R}^n$.

- $\{\vec{u}_1, \cdots \vec{u}_r\}$ is the orthonormal basis of Column (A)
- $\{\vec{u}_{r+1}, \cdots, \vec{u}_m\}$ is the orthonormal basis of left Nul space (A)

Column (A) $\perp$ left Nul

- $\{\vec{v}_1, \cdots \vec{v}_r\}$ is the orthonormal basis of row space (A)
- $\{\vec{v}_{r+1}, \cdots \vec{v}_n\}$ is the orthonormal basis of null space (A)

row (A) $\perp$ Nul (A)

- $$A = \lambda_1 \underbrace{\underset{m\times 1}{\vec{u}_1}\,\underset{1\times n}{\vec{v}_1^T}}_{\substack{m\times n \text{ matrix} \\ \text{rank } 1}} + \lambda_2 \underbrace{\vec{u}_2\,\vec{v}_2^T}_{\substack{m\times n \\ \text{rank } 1}} + \cdots + \lambda_r \underbrace{\vec{u}_r\,\vec{v}_r^T}_{\substack{m\times n \\ \text{rank } 1.}}$$

- $$\vec{u}\cdot\vec{v}^T = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} u_1v_1 & u_1v_2 & \cdots & u_1v_n \\ u_2v_1 & u_2v_2 & \cdots & u_2v_n \\ \vdots & & & \\ u_mv_1 & u_mv_2 & \cdots & u_mv_n \end{bmatrix} \begin{matrix} u_1\vec{v}^T \\ u_2\vec{v}^T \\ \\ u_m\vec{v}^T \end{matrix}$$

$\quad v_1\vec{u} \quad v_2\vec{u} \quad v_n\vec{u}$

rank 1 matrix
row space $= \{v\}$
Column space $\{\vec{u}\}$

$$\vec{u}_1 = \begin{bmatrix} u_{11} \\ \vdots \\ u_{1m} \end{bmatrix}$$

$$A = \lambda_1 \vec{u}_1\vec{v}_1^T + \lambda_2 \vec{u}_2\vec{v}_2^T \Rightarrow$$

row space = span $\{\vec{v}_1, \vec{v}_2\}$
Column space = span $\{\vec{u}_1, \vec{u}_2\}$

- row space of A

$$\begin{bmatrix} u_{11}\vec{v}_1^T \\ u_{12}\vec{v}_1^T \\ \vdots \\ u_{1m}\vec{v}_1^T \end{bmatrix} + \begin{bmatrix} u_{21}\vec{v}_2^T \\ u_{22}\vec{v}_2^T \\ \vdots \\ u_{2m}\vec{v}_2^T \end{bmatrix} = \begin{bmatrix} u_{11}\vec{v}_1^T + u_{21}\vec{v}_2^T \\ u_{12}\vec{v}_1^T + u_{22}\vec{v}_2^T \\ \vdots \\ u_{1m}\vec{v}_1^T + u_{2m}\vec{v}_2^T \end{bmatrix}$$

$\rightarrow$ linear Combination of $\vec{v}_1$ and $\vec{v}_2$

- $\mathrm{Nul}(AA^T) = \mathrm{Nul}(A^T)$

$$AA^T = U\Sigma\Sigma^T U^T \qquad \Sigma\Sigma^T = \begin{bmatrix} \lambda_1^2 & & & & \\ & \ddots & & & \\ & & \lambda_r^2 & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix}$$

eigen vector u_1 eigenvalue λ_1^2 $\Rightarrow$ $AA^T\vec{u}_1 = \lambda_1^2\vec{u}_1$

u_2 eigenvalue λ_2^2 $\quad AA^T\vec{u}_2 = \lambda_2^2\vec{u}_2$

$\vdots$ $\quad\quad \vdots$

u_r eigenvalue λ_r^2 $\quad AA^T\vec{u}_r = \lambda_r^2\vec{u}_r$

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u_{r+1} eigenvalue 0 $\Rightarrow$ $AA^T\vec{u}_{r+1} = \vec{0}$

$\vdots$ $\quad\quad \vdots$

u_n eigenvalue 0 $\quad AA^T\vec{u}_n = \vec{0}$

$\Uparrow$

Find $n-r$ vectors as basis $\rightarrow$ $\vec{u}_{r+1} \cdots \vec{u}_n$ lies in the

$\dim(\mathrm{Nul}(AA^T)) = n-r$ $\rightarrow$ $\mathrm{Nul}(AA^T)$

$\quad\quad \| \quad\quad\quad\quad\quad\quad\quad\quad \Downarrow$

$\dim(\mathrm{Nul}(A^T))$

$$\mathrm{Nul}(AA^T) = \mathrm{span}\{\vec{u}_{r+1} \cdots \vec{u}_n\}$$

$\quad \|$

$\mathrm{Nul}(A^T)$

Find SVD of $A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$

- $A^T A = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$

$p(\lambda) = \det \begin{bmatrix} 25-\lambda & 7 \\ 7 & 25-\lambda \end{bmatrix} = (25-\lambda)^2 - 7^2 \rightarrow$ $25-\lambda = 7 \Rightarrow \lambda = 18$, $25-\lambda = -7 \Rightarrow \lambda = 32$

$\Sigma = \begin{bmatrix} \sqrt{32} & \\ & \sqrt{18} \end{bmatrix}$ $\quad V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

eigenvector of $A^T A$.

! remember to normalize your eigenvector to UNIT vector

How to compute $\vec{u}_1 = \frac{1}{\lambda_1} A \vec{v}_1$

$A = \lambda_1 \vec{u}_1 \vec{v}_1^T + \lambda_2 \vec{u}_2 \vec{v}_2^T$

$A\vec{v}_1 = \lambda_1 \vec{u}_1 \underbrace{\vec{v}_1^T \vec{v}_1}_{=1} + \lambda_2 \vec{u}_2 \underbrace{\vec{v}_2^T \vec{v}_1}_{=0} = \lambda_1 \vec{u}_1 \Rightarrow \vec{u}_1 = \frac{1}{\lambda_1} A \vec{v}_1$

$\{\vec{v}_1, \vec{v}_2\}$ is an orthonormal basis

$\vec{u}_1 = \frac{1}{\lambda_1} A \vec{v}_1 = \frac{1}{\sqrt{32}} \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\leftarrow$ fix the sign first

$\vec{u}_2 = \frac{1}{\lambda_2} A \vec{v}_2$ $\cdots\cdots$ $= \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{32} & \\ & \sqrt{18} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{-2} \end{bmatrix}$

If you compute $A^T A$ to get V, you shouldn't get U from AA^T

If you compute AA^T to get U, you shouldn't get V from $A^T A$

You can get $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -\sqrt{32} & \\ & \sqrt{18} \end{bmatrix} \cdots$

may $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{32} & \\ & -\sqrt{18} \end{bmatrix}$

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -\sqrt{32} & \\ & -\sqrt{18} \end{bmatrix}$

You can't determine the sign.

$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{2\times 4}$

You can compute AA^T 2×2

A^TA 4×4

$AA^T \rightarrow U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix}$

$v_1 = \frac{1}{\lambda_1} A^T u_1 = \begin{bmatrix} \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \\ 0 \end{bmatrix}$

$v_2 = \frac{1}{\lambda_2} A^T u_2 = \begin{bmatrix} 0 \\ \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix}$

How to compute $\vec{v}_3, \vec{v}_4$ (?)

$\text{span}\{\vec{v}_3, \vec{v}_4\} \perp \text{span}\{\vec{v}_1, \vec{v}_2\}$

$\Rightarrow$ V_3, V_4 is the orthogonal basis of $\text{Nul}\begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \end{bmatrix}$

$\begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \end{bmatrix} \vec{v}_3 = \begin{bmatrix} \vec{v}_1^T \vec{v}_3 \\ \vec{v}_2^T \vec{v}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

- compute basis of $\text{Nul}\begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix}$
- then use G-S to orthogonalize the basis
- normalize to unit vector

$$A^TA \Rightarrow \quad V = \begin{bmatrix} \sqrt{2}/2 & 0 & -\sqrt{2}/2 & 0 \\ 0 & \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ \sqrt{2}/2 & 0 & +\sqrt{2}/2 & 0 \\ 0 & \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{bmatrix}$$

remember to transpose

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 & 0 \\ 0 & \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ -\sqrt{2}/2 & 0 & +\sqrt{2}/2 & 0 \\ 0 & -\sqrt{2}/2 & 0 & \sqrt{2}/1 \end{bmatrix}$$

Ex. Column space $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\rightarrow$ let Nul ϕ

row space

$\rightarrow$ Nul