Recap and Fundamental Theorem of Linear Algebra.

same size

$Col(A) \subseteq \mathbb{R}^m$. span{ all n Column Vectors }

$Row(A) \subseteq \mathbb{R}^n$ span{ all m row Vectors }

same size

$Nul(A) \subseteq \mathbb{R}^n$ $\{x \mid$ all Solution of $Ax=0\}$

$Nul(A^T) \subseteq \mathbb{R}^m$ $\{y \mid$ all Solution of $A^Ty=0\}$ $= y^TA=0$

left Null space "left Nul space"

$(A^Ty)^T = y^TA$

REF $\begin{cases} \dim[Col(A)] = r \\ \dim[Row(A)] = r \end{cases}$ add size m add given n

#Free Variable $\dim(Nul(A)) = n-r$

$\dim(Nul(A^T)) = m-r$

$A \in \mathbb{R}^{m\times n}$. rank$(A) = r$.

m rows and n columns

$Ax=b$ have m Eq n Variable

$A^T \in \mathbb{R}^{n\times m}$. $A^Ty=b$ have n Eq and m Variable.

Geometry Meaning of

$$\dim[Row(A)] + \dim(Nul(A)) = n$$

The Solution $\vec{x}$ should orthogonal to all row Vector

Let's write down A using row Representation

$$Ax = 0 \iff \underbrace{\begin{bmatrix} \vec{r}_1^T \\ \vec{r}_2^T \\ \vdots \\ \vec{r}_m^T \end{bmatrix} \vec{x}}_{Ax =} = \begin{bmatrix} \vec{r}_1\cdot\vec{x} \\ \vec{r}_2\cdot\vec{x} \\ \vdots \\ \vec{r}_m\cdot\vec{x} \end{bmatrix} = 0$$

$\vec{x} \in \mathbb{R}^n$

$\vec{r}_1\cdot\vec{x} = 0$ $\vec{r}_1 \perp \vec{x}$

$\vec{r}_2\cdot\vec{x} = 0$ $\vec{r}_2 \perp \vec{x}$

$\vdots$

$\vec{r}_m\cdot\vec{x} = 0$ $\vec{r}_m \perp \vec{x}$

$\vec{r}_1 \cdots \vec{r}_m$ all all row vectors. $\vec{r}_i \in \mathbb{R}^n$!!

$$Nul(A) = (Row(A))^{\perp}$$

$Nul(A) = \{x \mid Ax=0\} = \{x \mid$ all vectors that is orthogonal to the row vectors $\}$

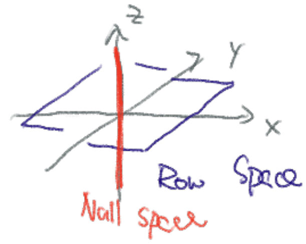
Row Space

Example if $Row(A), Nul(A) \subseteq \mathbb{R}^3$ $n=3$

$$A = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow x\text{-axis} \\ \leftarrow y\text{-axis} \end{array}$$

all solution of $Ax=0$ is $x = \left\{ \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} \middle| \, x_3 \in \mathbb{R} \right\}$

$\begin{cases} Row(A) = x\text{-}y \text{ plane} \\ Nul(A) = z\text{-axis} \end{cases}$



Similarly. $Nul(A^T) = (Col(A))^{\perp}$

Linear Algebra

Midterm Review Question

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Exercise Consider the matrix:

$$A = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 3 & 5 & 1 & 9 \end{bmatrix}$$

Elimination!

- Write down $A = LU$ where L is an lower traingular matrix and U is a REF. Same as $L\ U \to$ REF
- Calculate the four fundamental subspaces

$$A \to \underbrace{\begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}}_{E_{31} A} R3 - R1 \to \underbrace{\begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{E_{32}(E_{31}A) = E_{32} E_{31} A} R3 - R2$$

REF $= U$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad E_{32} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix}$$

$$E_{32} E_{31} A = U \qquad A = (E_{32} E_{31})^{-1} U = E_{31}^{-1} E_{32}^{-1} A$$

be careful about the order!

Third.

↑

→ Eliminate Second

first Eliminate first col

$$E_{31}^{-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ 1 & & 1 \end{bmatrix} \qquad E_{32}^{-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix}$$

$$L = E_{31}^{-1} E_{32}^{-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ 1 & 1 & 1 \end{bmatrix}$$

!! You should follow My order of Elimination Method if you want to copy it.

$$L = \begin{bmatrix} 1 & & \\ & 1 & \\ 1 & 1 & 1 \end{bmatrix}, \quad U = \begin{array}{c} \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \quad\ \ x_2 \ \ x_3 \qquad\ x_5 \end{array} \qquad A = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 3 & 5 & 1 & 9 \end{bmatrix}$$

Make sure to check whether $LU = A$

1. $\mathrm{Row}(A) = \mathrm{Row}(U) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 5 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

Copy the pivot rows !!!

2. $\mathrm{Nul}(A) = \mathrm{Nul}(U)$

x_2, x_3, x_5 are free

Solve x_1, x_4

First Solve the Eq

- $x_1 = -3x_2 - 5x_3 - 7x_5$
- $x_4 = -2x_5$

Then write down in vector form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -3x_2 - 5x_3 - 7x_5 \\ x_2 \\ x_3 \\ -2x_5 \\ x_5 \end{pmatrix}$$

free

$$= x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -7 \\ 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

Exercise 1. all the possible rank of

$$\begin{bmatrix} 1 & 1 & a \\ 3 & 3 & a \\ a & a & a \end{bmatrix}$$

← by your self

when a varies.

2. all the possible rank of

$$\begin{bmatrix} 1 & 3 & a \\ 3 & 1 & a \\ a & a & a \end{bmatrix}$$

when a varies.

Elimination

$$\begin{pmatrix} 1 & 3 & a \\ 3 & 1 & a \\ a & a & a \end{pmatrix} \xrightarrow[R3 \leftarrow R3 - aR1]{R2 \leftarrow R2 - 3R1} \begin{pmatrix} 1 & 3 & a \\ 0 & -8 & -2a \\ 0 & -2a & a-a^2 \end{pmatrix}$$

← don't divide by zero.

check if this will be 0 here $8 \neq 0$

$$\xrightarrow{R3 \leftarrow R3 - \frac{a}{4}R1} \begin{pmatrix} 1 & 1 & a \\ 0 & -8 & -2a \\ 0 & 0 & a - \frac{a^2}{2} \end{pmatrix}$$

$a=0$, or $a=1/2$

if $a - \frac{a^2}{2} = 0$, then rank $= 2$

otherwise rank $\neq 3$

$a \neq 0$, and $a \neq 1/2$

☆ if you set

$$\begin{pmatrix} 1 & 1 & a \\ & a & -2a \\ & & a - \frac{a^2}{2} \end{pmatrix}$$

then $a = 0 \Rightarrow$ rank $= 1$

$a = 1/2 \Rightarrow$ rank $= 2$.

otherwise $\Rightarrow$ rank $= 3$

Exercise 1. What is all the possible rank of

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

when a, b, c, d varies.

2. When is A invertible?

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \to \quad E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \to \quad E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ a & b & c & d \end{bmatrix}$$

$$E_{41} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \quad \to \quad E_{41}E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \to \quad E_{32}E_{41}E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & b-a & c-a & d-a \end{bmatrix}$$

$$E_{42} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad \to \quad E_{42}E_{32}E_{41}E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{bmatrix}$$

$$E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad \to \quad E_{43}E_{42}E_{32}E_{41}E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

asking number of 0

in a, $b-a$, $c-b$, $d-c$

$\swarrow$ calculating Col $\qquad \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \text{span}\{\cdots\}$

Exercise For which $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ are there solutions to $Ax = b$, where the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}$?

For those b, write down the complete solution.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 2 & 2 & 3 & b_2 \\ 0 & 0 & 1 & b_3 \end{array}\right) \xrightarrow{R2 \leftarrow R2 - 2R1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & 1 & b_3 \end{array}\right)$$

$$\xrightarrow{R3 \leftarrow R3 - R2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - (b_2 - 2b_1) \end{array}\right)$$

$= b_3 + 2b_1 - b_2$

$! \; b_3 + 2b_1 - b_2 = 0 \Rightarrow b_3 = b_2 - 2b_1$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

x_2 is free

Solve x_1, x_3

$x_1 = -x_2 - b_2 + 3b_1$

$x_3 = b_2 - 2b_1$

Exercise Calculate the inverse matrix of $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 4 \end{pmatrix}$?

Use elmination start from $[M|I]$ to $[I|M^{-1}]$

$$
\begin{aligned}
[M|I] = \begin{pmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 1 & 2 & 2 & | & 0 & 1 & 0 \\ 1 & 3 & 4 & | & 0 & 0 & 1 \end{pmatrix} & \xrightarrow[R3 \ \leftarrow R3 - R1]{R2 \ \leftarrow R2 - R1} \begin{pmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & -1 & 1 & 0 \\ 0 & 2 & 3 & | & -1 & 0 & 1 \end{pmatrix} \\
& \text{Use R1 to ellimate the column 1 in R2 and R3} \\
& \xrightarrow[R3 \ \leftarrow R3 - 2 \cdot R2]{R1 \ \leftarrow R1 - 1 \cdot R2} \begin{pmatrix} 1 & 0 & 0 & | & 2 & -1 & 0 \\ 0 & 1 & 1 & | & -1 & 1 & 0 \\ 0 & 0 & 1 & | & 1 & -2 & 1 \end{pmatrix} \\
& \text{Use R2 to ellimate the column 2 in R1 and R3} \\
& \xrightarrow[R2 \ \leftarrow R2 - 1 \cdot R3]{R1 \ \leftarrow R1 - 0 \cdot R3} \begin{pmatrix} 1 & 0 & 0 & | & 2 & -1 & 0 \\ 0 & 1 & 0 & | & -2 & 3 & -1 \\ 0 & 0 & 1 & | & 1 & -2 & 1 \end{pmatrix} \\
& \text{Use R3 to ellimate the column 3 in R1 and R2}
\end{aligned} \tag{1}
$$

Thus

$$M^{-1} = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{pmatrix}$$

Check if MM^{-1} is idenity! equal to check

- $(1,1,1) \cdot (2,-2,1) = 1$,$(1,2,2) \cdot (2,-2,1) = 0$, $(1,3,4) \cdot (2,-2,1) = 0$
- $(1,1,1) \cdot (-1,3,-2) = 0$,$(1,2,2) \cdot (-1,3,-2) = 1$, $(1,3,4) \cdot (-1,3,-2) = 0$
- $(1,1,1) \cdot (0,-1,1) = 0$,$(1,2,2) \cdot (0,-1,1) = 0$, $(1,3,4) \cdot (0,-1,1) = 1$

① x_1, x_2 free variables or $\mathrm{Nul}(A) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ $\dim(\mathrm{Nul}(A)) = 2$

② $A \in \mathbb{R}^{m \times n}$ $\qquad \dim(\mathrm{row}(A)) = \dim(\mathrm{col}(A)) = n - \underbrace{\quad}_{n - \dim(\mathrm{Nul}(A))} = 5 - 2 = 3$

$n = 5$ # Variables

but I may not know m # Eq

$\dim(\mathrm{row}(A)) =$

1. The complete solution of linear system $Ax = b$ is $\vec{x} = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 1 \\ 2 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, then $\dim(\mathrm{col}(A)) = 3$

Yes, A have 5 column ($n = 5$). For we have 2 free variables, thus $\dim(\mathrm{Nul}(A)) = 2$ So rank $r = n - \dim(\mathrm{Nul}(A)) = 5 - 2 = 3$

We don't know $\dim(\mathrm{nul}(A^T))$ because we don't know m

2. There exist a matrix A whose column space is spanned by $(1, 2, 3)$ and $(1, 0, 1)$and whose nullspace is spanned by $(1, 2, 3, 6)$ $\mathbb{R}^4 \Rightarrow n = 4$ $\qquad \mathbb{R}^3 \Rightarrow m = 3$

No. The dimensions of such a matrix must be 3 by 4 ($m = 3$ and $n = 4$). The dimension of the column space is 2, because the given vectors are independent. That means the dimension of the nullspace must be $4 - 2 = 2$. The null space cannot be spanned by 1 vector.

$A \in \mathbb{R}^{m \times n}$ $\qquad A \in \mathbb{R}^{3 \times 4}$ $\qquad \dim(\mathrm{col}(A)) = 2$ $\qquad$ by Fundamental Thm

$\dim(\mathrm{nul}(\mathrm{space})) = 1$ $\qquad \underbrace{\dim(\mathrm{col})}_{2} + \underbrace{\dim(\mathrm{nul})}_{1} = n = 4$

3.

- For a matrix $A \in \mathbb{R}^{4 \times 5}$, the largest possible rank of A is 5. No, rank$\leq m$, rank $\leq n$
- For a matrix $A \in \mathbb{R}^{4 \times 5}$ there are possibility that linear system $Ax = b$ have one and only have one solution. No, rank ≤ 4, so this can't be a full column rank matrix, $\dim(\mathrm{Nul}(A))$ can't be zero. Futhermore, this linear sytem must have infinite solution. Because this is a full row rank matrix, thus the system at least have one solution.($\mathrm{row}(A) = \mathbb{R}^4$)
- For a matrix $A \in \mathbb{R}^{4 \times 5}$, $\mathrm{rank}(A) = 4$. There are possibility that linear system $Ax = b$ have one and only have one solution. No, same as above. $\dim(\mathrm{Nul}(A)) = 1$
- For a matrix $A \in \mathbb{R}^{4 \times 3}$, $\mathrm{rank}(A) = 3$. There are possibility that linear system $Ax = b$ have one and only have one solution. Yes, this is a full column rank matrix, so $\mathrm{Nul}(A) = \{\vec{0}\}$ $\Rightarrow$ must have infinite solution!
- For a matrix $A \in \mathbb{R}^{4 \times 5}$, $\mathrm{rank}(A) = 4$. There are possibility that linear system $Ax = b$ have no solution. No. This is a full row rank matrix, so $\dim(\cancel{\mathrm{row}}\,\mathrm{col}(A)) = 4$ and $\cancel{\mathrm{row}}\,\mathrm{col}(A) \subset \mathbb{R}^4$. This means $\cancel{\mathrm{row}}\,\mathrm{col}(A) = \mathbb{R}^4$. Every system $Ax = b$ must have a solution. $\dim(\mathrm{Nul}(A)) = 1 \Rightarrow$ have 0 or infinite solution.
- For a matrix $A \in \mathbb{R}^{5 \times 4}$, $\mathrm{rank}(A) = 4$. There are possibility that linear system $Ax = b$ have no solution. Yes, this is because this is not a full row rank matrix.
- $Y = AX$ and A is an invertible matrix, then $\mathrm{rank}(Y) = \mathrm{rank}(X)$. Yes,because $Y = AX$ so $\mathrm{rank}(Y) \leq \mathrm{rank}(X)$. For A is invertible matrix, so $X = A^{-1}Y$ which tells us $\mathrm{rank}(X) \leq \mathrm{rank}(Y)$. The only possiblity is $\mathrm{rank}(Y) = \mathrm{rank}(X)$

$X = [\vec{v}_1 \cdots \vec{v}_n]$ $\qquad \mathrm{rank}(AB) \leq \mathrm{rank}(A)$

$Y = [A\vec{v}_1 \cdots A\vec{v}_n]$ $\qquad \mathrm{rank}(AB) \leq \mathrm{rank}(B)$ $\qquad \Rightarrow \mathrm{rank}(X) = \mathrm{rank}(Y)$

$\mathrm{rank}(X) = \mathrm{rank}(A^{-1}Y) \leq \mathrm{rank}(Y)$

Harder Question:

$Nul(A^T) = m - r$ needs more information!

What is all the possible Value?

$m \geq r$, $\dim(Nul(A^T))$ can be any $\mathbb{Z}^+$

2# Free Variable $\Rightarrow \dim[Nul(A)] = 2$

1. The complete solution of linear system $Ax = b$ is $\vec{x} = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 1 \\ 2 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, then $\dim(\text{col}(A)) = 3$

$A \in \mathbb{R}^{m \times n}$? what is m? we need more information

m is # Eq

what is n? $n = 5$

n is # Variable

$= \text{rank } A = 3$

$\Rightarrow \dim(Col(A)) = n - \#\text{Free Variable} = 5 - 2 = 3$, $\dim(Row(A)) = \dim(Col(A))$

2. There exist a matrix A whose column space is spanned by $(1, 2, 3)$ and $(1, 0, 1)$ and whose nullspace is spanned by $(1, 2, 3, 6)$

Fix the Size of A $n = 4$ $m = 3$

$\dim(Col(A)) = 2$

$\dim(Nul(A)) = 1$

be careful

is this possible? No! $\dim(Col(A)) + \dim(Row(A)) = n$

but $2 + 1 \neq 4$!!!

3.

- For a matrix $A \in \mathbb{R}^{4\times5}$, the largest possible rank of A is 5. No $\text{rank}(A) \leq 4$
- For a matrix $A \in \mathbb{R}^{4\times5}$ there are possibility that linear system $Ax = b$ have one and only have one solution. No
- For a matrix $A \in \mathbb{R}^{4\times5}$, $\text{rank}(A) = 4$. There are possibility that linear system $Ax = b$ have one and only have one solution. No
- For a matrix $A \in \mathbb{R}^{4\times3}$, $\text{rank}(A) = 3$. There are possibility that linear system $Ax = b$ have one and only have one solution. Yes
- For a matrix $A \in \mathbb{R}^{5\times4}$, $\text{rank}(A) = 4$. There are possibility that linear system $Ax = b$ have one and only have one solution. Yes
- For a matrix $A \in \mathbb{R}^{4\times5}$, $\text{rank}(A) = 4$. There are possibility that linear system $Ax = b$ have no solution. No
- For a matrix $A \in \mathbb{R}^{5\times4}$, $\text{rank}(A) = 4$. There are possibility that linear system $Ax = b$ have no solution. Yes
- $Y = AX$ and A is an invertible matrix, then $\text{rank}(Y) = \text{rank}(X)$. Yes

$[Av_1 \cdots Av_n]$ $[v_1 \cdots v_n]$

We know $\text{rank}(AB) \leq \text{rank}(A)$, $\text{rank}(AB) \leq \text{rank}(B)$

Firstly $\text{rank}(AX) \leq \text{rank}(X)$

$Y = AX$ and A is invertible means.

$A^{-1}Y = X$ so $\text{rank}(X) = \text{rank}(A^{-1}Y) \leq \text{rank}(Y)$

$\Rightarrow$ Combine $\text{Rank}(Y) = \text{Rank}(X)$

Checking linear Subspace.

1. any $V_1 . V_2 \in V$ and $C_1 , C_2 \in \mathbb{R}$

we have $C_1V_1 + C_2V_2 \in V$

Ex. $V = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a.b.c.d \in \mathbb{R} , \; a+b+c+d=0 \right\}$

is a vector space.

$V_1 . V_2 \in V$ then $C_1V_1 + C_2V_2 \in V$

checking $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ $a_1+b_1+c_1+d_1=0$

$\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ $a_2+b_2+c_2+d_2=0$

then

$\Rightarrow \begin{pmatrix} x_1a_1+x_2a_2 . & x_1b_1+x_2b_2 \\ x_1c_1+x_2c_2 , & x_1d_1+x_2d_2 \end{pmatrix}$

satisfies

$(x_1a_1+x_2a_2)+(x_1b_1+x_2b_2)$
$+(x_1c_1+x_2c_2)+(x_1d_1+x_2d_2)$
$=0$

---

a . b . c. are free variables

means once a. b. c are fixed. then d is fixed.

(you can also understand
- b.c.d are free. then a fixed.
- a.c.d are free b fixed
- a.b. d are free c fixed)

- Find basis : set one of free variable to 1
↓ all the other to zero

- $a=1$ $b=0$ $c=0$ $\Rightarrow d=-1 \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- $a=0$ $b=1$ $c=0$ $\Rightarrow d=-1 \Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$
- $a=0$ $b=0$ $c=1$ $\Rightarrow d=-1 \Rightarrow \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$

basis

dim = 3 !!

---

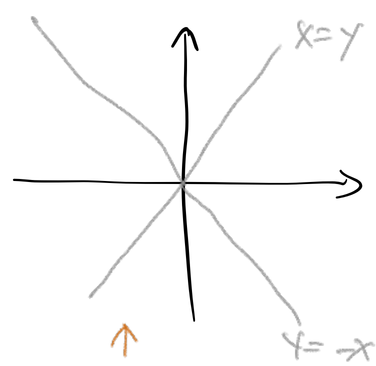
is $V = \{ (x, y) \mid x^2 - y^2 = 0 \}$ a Vector space

To verify a set is not a vector space. you only to give an example.

$(1. 1) \in V$ because $1^2 - 1^2 = 0$

$(1. -1) \in V$ $1^2 - (-1)^2 = 0.$

but $(2, 0) = (1.1) + (1, -1) \notin V$.



This is how V look like