Linear Algebra

Midterm Review Question

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January 2024

Exercise Consider the matrix:

$$A = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 3 & 5 & 1 & 9 \end{bmatrix}$$

- Write down $A = LU$ where L is an lower traingular matrix and U is a REF.
- Calculate the four fundamental subspaces

Exercise 1. all the possible rank of

$$\begin{bmatrix} 1 & 1 & a \\ 3 & 3 & a \\ a & a & a \end{bmatrix}$$

when a varies.

2. all the possible rank of

$$\begin{bmatrix} 1 & 3 & a \\ 3 & 1 & a \\ a & a & a \end{bmatrix}$$

when a varies.

Exercise 1. What is all the possible rank of

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

when a, b, c, d varies.

2. When is A invertible?

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \rightarrow \quad E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \rightarrow \quad E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ a & b & c & d \end{bmatrix}$$

$$E_{41} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \quad \rightarrow \quad E_{41}E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \rightarrow \quad E_{32}E_{41}E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & b-a & c-a & d-a \end{bmatrix}$$

$$E_{42} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad \rightarrow \quad E_{42}E_{32}E_{41}E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{bmatrix}$$

$$E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad \rightarrow \quad E_{43}E_{42}E_{32}E_{41}E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

Exercise For which $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ are there solutions to $Ax = b$, where the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}$?
For those b, write down the complete solution.

Exercise Calculate the inverse matrix of $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 4 \end{pmatrix}$?

Use elmination start from $[M|I]$ to $[I|M^{-1}]$

$$
\begin{aligned}
[M|I] = \begin{pmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 1 & 2 & 2 & | & 0 & 1 & 0 \\ 1 & 3 & 4 & | & 0 & 0 & 1 \end{pmatrix} &\xrightarrow[R3 \;\leftarrow R3 - R1]{R2 \;\leftarrow R2 - R1} \begin{pmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & -1 & 1 & 0 \\ 0 & 2 & 3 & | & -1 & 0 & 1 \end{pmatrix} \\
&\text{Use R1 to ellimate the column 1 in R2 and R3} \\
&\xrightarrow[R3 \;\leftarrow R3 - 2 \cdot R2]{R1 \;\leftarrow R1 - 1 \cdot R2} \begin{pmatrix} 1 & 0 & 0 & | & 2 & -1 & 0 \\ 0 & 1 & 1 & | & -1 & 1 & 0 \\ 0 & 0 & 1 & | & 1 & -2 & 1 \end{pmatrix} \\
&\text{Use R2 to ellimate the column 2 in R1 and R3} \\
&\xrightarrow[R2 \;\leftarrow R2 - 1 \cdot R3]{R1 \;\leftarrow R1 - 0 \cdot R3} \begin{pmatrix} 1 & 0 & 0 & | & 2 & -1 & 0 \\ 0 & 1 & 0 & | & -2 & 3 & -1 \\ 0 & 0 & 1 & | & 1 & -2 & 1 \end{pmatrix} \\
&\text{Use R3 to ellimate the column 3 in R1 and R2}
\end{aligned} \tag{1}
$$

Thus

$$
M^{-1} = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{pmatrix}
$$

Check if MM^{-1} is idenity! equal to check

- $(1,1,1) \cdot (2,-2,1) = 1$, $(1,2,2) \cdot (2,-2,1) = 0$, $(1,3,4) \cdot (2,-2,1) = 0$
- $(1,1,1) \cdot (-1,3,-2) = 0$, $(1,2,2) \cdot (-1,3,-2) = 1$, $(1,3,4) \cdot (-1,3,-2) = 0$
- $(1,1,1) \cdot (0,-1,1) = 0$, $(1,2,2) \cdot (0,-1,1) = 0$, $(1,3,4) \cdot (0,-1,1) = 1$

1. The complete solution of linear system $Ax = b$ is $\vec{x} = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 1 \\ 2 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, then $\dim(\text{col}(A)) = 3$

2. There exist a matrix A whose column space is spanned by $(1, 2, 3)$ and $(1, 0, 1)$and whose nullspace is spanned by $(1, 2, 3, 6)$

3.

- For a matrix $A \in \mathbb{R}^{4\times 5}$, the largest possible rank of A is 5. No
- For a matrix $A \in \mathbb{R}^{4\times 5}$ there are possibility that linear system $Ax = b$ have one and only have one solution. No
- For a matrix $A \in \mathbb{R}^{4\times 5}$, $\text{rank}(A) = 4$. There are possibility that linear system $Ax = b$ have one and only have one solution. No
- For a matrix $A \in \mathbb{R}^{4\times 3}$, $\text{rank}(A) = 3$. There are possibility that linear system $Ax = b$ have one and only have one solution. Yes
- For a matrix $A \in \mathbb{R}^{5\times 4}$, $\text{rank}(A) = 4$. There are possibility that linear system $Ax = b$ have one and only have one solution. Yes
- For a matrix $A \in \mathbb{R}^{4\times 5}$, $\text{rank}(A) = 4$. There are possibility that linear system $Ax = b$ have no solution. No
- For a matrix $A \in \mathbb{R}^{5\times 4}$, $\text{rank}(A) = 4$. There are possibility that linear system $Ax = b$ have no solution. Yes
- $Y = AX$ and A is an invertible matrix, then $\text{rank}(Y) = \text{rank}(X)$. Yes