Sec 6.1:

1. Find the eigenvalues and eigenvectors of the following matrix

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 4 \end{bmatrix}.$$

Check that indeed $A\boldsymbol{x} = \lambda\boldsymbol{x}$ for each eigenvalue.

The characteristic polynomial is $(1-\lambda)(4-\lambda)$. Hence A has two eigenvalues 1 and 4. Write $\boldsymbol{x} = (x_1, x_2)$. Then, to compute the eigenvectors, we solve for $\boldsymbol{x}$ in $A\boldsymbol{x} = \lambda\boldsymbol{x}$, i.e., we solve $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$. That is,

$$\begin{bmatrix} 1-\lambda & 0 \\ -1 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This gives:

$$(1-\lambda)x_1 = 0, \tag{1}$$
$$-x_1 + (4-\lambda)x_2 = 0. \tag{2}$$

First, we set $\lambda = 1$. From (2), we get $3x_2 = x_1$. Hence, $(3, 1)$ is an eigenvector associated with the eigenvalue 1.

Let us set $\lambda = 4$. From (1), we can see $x_1 = 0$ and there is no requirement on x_2. So, $(0, 1)$ is an eigenvector associated with the eigenvalue 4.

Now let us check the results. We compute

$$\begin{bmatrix} 1 & 0 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 3 \\ 1 \end{bmatrix},$$

and

$$\begin{bmatrix} 1 & 0 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

2. Compute the eigenvalues and eigenvectors of the following rank one matrix:

$$\begin{bmatrix}1\\2\\1\end{bmatrix}\begin{bmatrix}2 & 1 & 2\end{bmatrix} = \begin{bmatrix}2 & 1 & 2\\4 & 2 & 4\\2 & 1 & 2\end{bmatrix}$$

What do you notice about the eigenvalues and eigenvectors? Explain.

Denote the above matrix by A. Set $\boldsymbol{u} = \begin{bmatrix}1\\2\\1\end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix}2\\1\\2\end{bmatrix}$ Then $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$. We can compute that A has two distinct eigenvalues. The first one is 6 with eigenvector $\boldsymbol{x}_1 = \begin{bmatrix}1\\2\\1\end{bmatrix}$. The second one is 0 with eigenvectors $\boldsymbol{x}_2 = \begin{bmatrix}-1\\2\\0\end{bmatrix}$ and $\boldsymbol{x}_3 = \begin{bmatrix}-1\\0\\1\end{bmatrix}$.

The observations are:

- $\boldsymbol{x}_1$ is proportional to $\boldsymbol{u}$, because for any vector $\boldsymbol{x}$, we must have that

$$A\boldsymbol{x} = \boldsymbol{u}\boldsymbol{v}^T\boldsymbol{x} = \boldsymbol{u}(\boldsymbol{v}\cdot\boldsymbol{x}) = (\boldsymbol{v}\cdot\boldsymbol{x})\boldsymbol{u}. \tag{3}$$

- $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ are orthogonal to $\boldsymbol{v}$, because in order to have $A\boldsymbol{x} = \boldsymbol{0}$, by (3), we must have $\boldsymbol{u}\boldsymbol{v}^T\boldsymbol{x} = \boldsymbol{0} \implies (\boldsymbol{v}^T\boldsymbol{x})\boldsymbol{u} = \boldsymbol{0} \implies \boldsymbol{v}^T\boldsymbol{x} = 0$, i.e., $\boldsymbol{v}\cdot\boldsymbol{x} = 0$, namely, that $\boldsymbol{x}$ is orthogonal to $\boldsymbol{v}$.
- The eigenvalue 0 gives two linearly independent eigenvectors, which must mean that this eigenvalue is repeated. In fact, since the matrix is 3×3 with rank 1, we can infer that the matrix is not invertible, i.e., it has determinant equal to zero, which means that at least one of the eigenvalues must be zero. We will later learn that the rank will tell us exactly how many zero eigenvalues there are, but that will have to wait till chapter 7.

3. Let A be a 3×3 matrix with linearly independent eigenvectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ corresponding to eigenvalues 0, 3, and 5, respectively.

   (a) Give a basis for the nullspace of A.
   A basis for the nullspace is given by $\{\boldsymbol{u}\}$.

   (b) Give a basis for the column space of A.
   A basis for the column space is given by $\{\boldsymbol{v}, \boldsymbol{w}\}$.

   (c) Find a vector $\boldsymbol{x}$, such that $A\boldsymbol{x} = \boldsymbol{v} + \boldsymbol{w}$.
   Since $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ form a basis for $\mathbb{R}^3$, we can express $\boldsymbol{x} = a\boldsymbol{u} + b\boldsymbol{v} + c\boldsymbol{w}$ for real numbers a, b, c to be determined. Now, we can compute

$$A\boldsymbol{x} = A(a\boldsymbol{u} + b\boldsymbol{v} + c\boldsymbol{w}) = aA\boldsymbol{u} + bA\boldsymbol{v} + cA\boldsymbol{w} = 3b\boldsymbol{v} + 5c\boldsymbol{w}.$$

   Hence, to have $A\boldsymbol{x} = \boldsymbol{v} + \boldsymbol{w}$, we must have

$$(3b - 1)\boldsymbol{v} + (5c - 1)\boldsymbol{w} = \boldsymbol{0}.$$

   The linear independence implies that $b = 1/3$ and $c = 1/5$. There is no condition for a. For convenience, we set $a = 0$. So, the vector $\boldsymbol{x}$ can be given be

$$\boldsymbol{x} = \frac{1}{3}\boldsymbol{v} + \frac{1}{5}\boldsymbol{w}.$$

4. Find three 2×2 matrices with trace 9 and determinant 20. What are the eigenvalues of each of these matrices?

   Some examples are

$$\begin{bmatrix} 1 & 3 \\ -4 & 8 \end{bmatrix}, \quad \begin{bmatrix} 2 & 2 \\ -3 & 7 \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 \\ -2 & 6 \end{bmatrix}.$$

   Now, let us compute the eigenvalues. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be such a matrix. Then, we have $a + d = 9$ and $ad - bc = 20$. The characteristic polynomial is given be

$$(a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + ad - bc = \lambda^2 - 9\lambda + 20.$$

   Then, we can compute that the eigenvalues are 4 and 5.

Sec 6.2:

1. Which of these matrices cannot be diagonalized?

$$A = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$$

The matrix A is not diagonalizable. We can compute that A has only one eigenvalue $\lambda = 0$ with algebraic multiplicity 2. However, it only has one independent eigenvector $(1, 1)$. So the geometric multiplicity of $\lambda = 0$ is 1. Since the two multiplicities are not equal, the matrix is not diagonalizable.

The matrix B is diagonalizable. We can compute that B has two distinct eigenvalues 2 and -2. Since B is 2×2, we must have that both multiplicities must be 1. This implies diagonalizability.

The last matrix C is not diagonalizable. We can compute that it has only one eigenvalue $\lambda = 2$ with algebraic multiplicity 2. However, it only has one independent eigenvector $(0, 1)$. Hence, the geometric multiplicity of $\lambda = 2$ is 1 not equal to the algebraic multiplicity. Therefore, it is not diagonalizable.

2. Suppose A is a 3×3 matrix with eigenvalues 1, 2, and 4. What is the trace of A^2? What is the determinant of A^{-1}?

We know that the square of an eigenvalue of A is an eigenvalue of A^2. Hence, $1, 4, 16$ are eigenvalues of A^2. Since A is 3×3, we conclude that these are exactly the eigenvalues of A^2, and each of them has (both algebraic and geometric) multiplicity 1. Since the trace of A^2 is the sum of eigenvalues, we deduce that the trace of A^2 is 21.

The determinant of A is $1 \times 2 \times 4 = 8$. Since the determinant of A^{-1} is the reciprocal of the determinant of A, we conclude that the determinant of A^{-1} is $\frac{1}{8}$.

3. Suppose A is a 3×3 matrix with eigenvalues 1, 1, and 2. Which of the following statements is certain to be true? Why?

   (i) A is invertible.

   (ii) A is diagonalizable.

   (iii) A is not diagonalizable.

It is certain that (i) is always true, because the determinant of A is $1 \times 1 \times 2 = 2$ which is nonzero. However, (ii) and (iii) are not certain to be true. Indeed, we can see that the algebraic multiplicity of the eigenvalue 1 is 2. If its geometric multiplicity is 2 then A is diagonalizable; otherwise, A is not diagonalizable.

4. If $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$, find A^{100} by diagonalizing A.

We can compute that A has eigenvalue 1 with eigenvector $(-1, 1)$, and eigenvalue 5 with eigenvector $(3, 1)$. Hence, we can deduce that A^{100} has eigenvalue 1 with eigenvector $(-1, 1)$, and eigenvalue 5^{100} with eigenvector $(3, 1)$. The inverse matrix of $\begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$ is $\begin{bmatrix} -\frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$. Note that, A can be diagonalized as

$$A = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

Therefore, we can express A^{100} as

$$A^{100} = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5^{100} \end{bmatrix} \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

5. Show that the matrices $A = \begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}$ are similar.

We can compute that both A and B have two distinct eigenvalues 1 and 3. Let $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$. We know that A and B are diagonalizable. Hence, there are invertible U and V such that

$$A = U\Lambda U^{-1},$$
$$B = V\Lambda V^{-1}.$$

Then, by setting $C = UV^{-1}$, we can see

$$A = CBC^{-1}.$$

Therefore, we conclude that A is similar to B.