Sec 5.1, 5.2:

1. Use the properties of determinants to compute the following:

(a) $\det\begin{bmatrix}0&0&0&2\\0&0&2&6\\0&1&2&2\\4&4&8&8\end{bmatrix} = \det\begin{bmatrix}4&4&8&8\\0&1&2&2\\0&0&2&6\\0&0&0&2\end{bmatrix} = \det\begin{bmatrix}4&4&8&8\\0&1&2&2\\0&0&2&6\\0&0&0&2\end{bmatrix} = 4\cdot 1\cdot 2\cdot 2 = 16$

(b) $\det\begin{bmatrix}101&201&301\\102&202&302\\103&203&303\end{bmatrix} = \det\begin{bmatrix}101&201&301\\102&202&302\\1&1&1\end{bmatrix} = \det\begin{bmatrix}101&201&301\\1&1&1\\1&1&1\end{bmatrix} = 0$

(c) $\det\begin{bmatrix}1&2&-2&0\\2&3&-4&1\\-1&-2&0&2\\0&2&5&3\end{bmatrix} = \det\begin{bmatrix}1&2&-2&0\\0&-1&0&1\\-1&-2&0&2\\0&2&5&3\end{bmatrix} = \det\begin{bmatrix}1&2&-2&0\\0&-1&0&1\\0&0&-2&2\\0&2&5&3\end{bmatrix} = \det\begin{bmatrix}1&2&-2&0\\0&-1&0&1\\0&0&-2&2\\0&0&5&5\end{bmatrix}$
$= \det\begin{bmatrix}1&2&-2&0\\0&-1&0&1\\0&0&-2&2\\0&0&0&10\end{bmatrix} = 20$

2. True or False.

(a) If A and B are identical, except that $b_{11} = 2a_{11}$, then $\det B = 2\det A$.
False
Assuming $A = \begin{bmatrix}1&1\\1&1\end{bmatrix}$ and $B = \begin{bmatrix}2&1\\1&1\end{bmatrix}$, then, A and B are identical, except that $b_{11} = 2a_{11}$. However, $\det B = 1 \neq 2\det A = 0$

(b) The determinant is the product of the pivots.
False
Assuming $A = \begin{bmatrix}0&1\\1&0\end{bmatrix}$.
Then $\det A = -1$, but the two pivots are 1 and 1, so the product of the pivots is 1.

(c) If A is invertible and B is singular, then $A+B$ is invertible.
False
$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is invertible and $B = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$ is singular, but $A+B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is singular.

(d) If A is invertible and B is singular, then AB is singular.
True
$\det(AB) = \det A * \det B = 1 * 0 = 0$
Thus, AB is singular.

(e) The determinant of $AB - BA$ is zero.
False
Assuming $A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 \\ 5 & 0 \end{bmatrix}$, Thus, $AB - BA = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. The determinant of $AB - BA$ is not equal to zero.

Sec 5.3:

1. Use cofactor expansion to find the determinants of:

(a) $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Expand the first row and we find only the second cofactor will be kept:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The remaining cofactor is,

$$M_{43} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Thus, $\det A = \det \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = -\det \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Similarily,

$$\det A = \det \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = -\det \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$$

(b) $B = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Expand the last row and we find only the last cofactor will be kept:

$$\det B = \det \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The kept cofactor will be:

$$M_{44} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 3 & 4 \\ 6 & 7 & 8 \end{bmatrix}$$

Thus,

$$\det B = (-1)^{4+4} \cdot 1 \cdot \det M_{44} = \det \begin{bmatrix} 0 & 0 & 1 \\ 0 & 3 & 4 \\ 6 & 7 & 8 \end{bmatrix} = \det \begin{bmatrix} 0 & 3 \\ 6 & 7 \end{bmatrix} = -18$$

2. Consider the matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$.

(a) Compute the cofactors of A, and determine the cofactor matrix C.

$C_{11} = (-1)^{1+1} \cdot \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 3$

$C_{12} = (-1)^{1+2} \cdot \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix} = 2$

$C_{13} = (-1)^{1+3} \cdot \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix} = 1$

$C_{21} = (-1)^{2+1} \cdot \begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix} = 2$

$C_{22} = (-1)^{2+2} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 4$

$C_{23} = (-1)^{2+3} \cdot \begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix} = 2$

$C_{31} = (-1)^{3+1} \cdot \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix} = 1$

$C_{32} = (-1)^{3+2} \cdot \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix} = 2$

$C_{33} = (-1)^{3+3} \cdot \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 3$

Thus, $C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

(b) Use your result in (a) to compute A^{-1}.

$\det A = \det \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = -\det \begin{bmatrix} -1 & 2 & -1 \\ 2 & -1 & 0 \\ 0 & -1 & 2 \end{bmatrix} = -\det \begin{bmatrix} -1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$

$= \det \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} = 3 * 2 - (-2) * (-1) = 4$

$A^* = C^T = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

$A^{-1} = \frac{A^*}{\det A} = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$

3. A box has edges from $(0,0,0)$ to $(3,1,1)$, $(1,3,1)$, and $(1,1,3)$. Find the volume of the box.
The three edges from the same point of the box are:
$\boldsymbol{a} = (3,1,1)$
$\boldsymbol{b} = (1,3,1)$
$\boldsymbol{c} = (1,1,3)$

The volume of the box is:

$$V = \left| \det \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \right| = 20$$

4. The corners of a triangle are $(2,1)$, $(3,4)$, and $(0,5)$. What is the area?
The edge from the point $(2,1)$ to the point $(3,4)$ is:

$$\boldsymbol{a} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

The edge from the point $(0,5)$ to the point $(3,4)$ is:

$$\boldsymbol{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

Notice that $\boldsymbol{a} \perp \boldsymbol{b}$, so these form the base and hight of the triangle. Hence, the area of the triangle is:

$$\frac{1}{2} \left| \det \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} \right| = \frac{10}{2} = 5.$$