Sec 4.3:

1. Find the best-fit line $b = C + Dt$ through the points $(1,1)$, $(2,5)$, and $(-1,2)$.

   We want to find constants C, D such that $\sum_{i=1}^{3}(C + Dx_i - y_i)^2$ is minimized, which is the same as finding the least squares solution of

$$A\boldsymbol{x} = \boldsymbol{b}$$

   where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & -1 \end{bmatrix}, \boldsymbol{x} = \begin{bmatrix} C \\ D \end{bmatrix}, \text{ and } \boldsymbol{b} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}.$$

   And we know that the least squares solution is given by

$$\hat{\boldsymbol{x}} = (A^T A)^{-1} A^T \boldsymbol{b} = \begin{bmatrix} 2.1429 \\ 0.7857 \end{bmatrix}$$

   Thus the best fit line is $b = 2.1429 + 0.7857t$.

2. Find the projection of $(2, 3, -2, 1)$ onto the nullspace of $A = \begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix}$.

   We find the nullspace by solving $A\boldsymbol{x} = 0 \implies N(A) = \text{span}\{(0, 0, 2, -1), (3, -1, 0, 0)\}$. Let $\boldsymbol{b} = (2, 3, -2, 1)$. To find the projection of $\boldsymbol{b}$ onto $N(A)$, we minimize $||\boldsymbol{b} - B\boldsymbol{x}||^2$, where

$$B = \begin{bmatrix} 0 & 3 \\ 0 & -1 \\ 2 & 0 \\ -1 & 0 \end{bmatrix}$$

   Such optimal value $\hat{\boldsymbol{x}}$ is again the least squares solution

$$\hat{\boldsymbol{x}} = (B^T B)^{-1} B^T \boldsymbol{b} = \begin{bmatrix} -1 \\ 0.3 \end{bmatrix}$$

   and thus the projection is

$$B\hat{\boldsymbol{x}} = \begin{bmatrix} 0.9 \\ -0.3 \\ -2 \\ 1 \end{bmatrix}$$

Sec 4.4:

1. **Orthogonal matrices:** The following matrices are important examples of orthogonal matrices.

   (a) **Rotation matrix** Consider

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

   i. Show that this is indeed an orthogonal matrix.
   Check that $Q^TQ = I$.

   ii. Choose any θ you wish (e.g., $\pi/2$) and explain what happens when it acts on a vector $\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
   Substitute $\theta = \pi/2$ and compute

$$Qx = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

   Thus Q rotates x by $\pi/2$ counterclockwise.

(b) **Permutation matrix** Consider

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

i. Show that this is indeed an orthogonal matrix.
The column vectors have unit norms and are mutually perpendicular, so Q is an orthogonal matrix. Alternatively you can check that $Q^T Q = I$.

ii. What operation does this matrix perform?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ z \\ y \end{bmatrix}$$

switching the 2nd and the 3rd element of a vector of size 3.

(c) **Reflection matrix** Let $\boldsymbol{u}$ be any unit vector. Then

$$Q = I - 2\boldsymbol{u}\boldsymbol{u}^T$$

i. Show that this is indeed an orthogonal matrix.

$$\begin{aligned} Q^TQ &= (I - 2\boldsymbol{u}\boldsymbol{u}^T)^T(I - 2\boldsymbol{u}\boldsymbol{u}^T) \\ &= (I - 2\boldsymbol{u}\boldsymbol{u}^T)(I - 2\boldsymbol{u}\boldsymbol{u}^T) \\ &= I - 2\boldsymbol{u}\boldsymbol{u}^T - 2\boldsymbol{u}\boldsymbol{u}^T + 4\boldsymbol{u}\boldsymbol{u}^T\boldsymbol{u}\boldsymbol{u}^T \\ &= I - 2\boldsymbol{u}\boldsymbol{u}^T - 2\boldsymbol{u}\boldsymbol{u}^T + 4\boldsymbol{u}\boldsymbol{u}^T \\ &= I \end{aligned}$$

ii. Choose $\boldsymbol{u} = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$. Calculate the reflection matrix Q. Then, explain what happens when this matrix acts on a vector $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}\begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

and

$$Q\boldsymbol{x} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

thus Q reflects $\boldsymbol{x}$ across $-45°$ (about line $y = -x$).

2. Find a third column so that the matrix

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{14} & \\ 1/\sqrt{3} & 2/\sqrt{14} & \\ 1/\sqrt{3} & -3/\sqrt{14} & \end{bmatrix}$$

is orthogonal.

Step 1. Find any vector $\boldsymbol{v} = (x, y, z)$ perpendicular to the first two columns.

$$x + y + z = 0$$
$$x + 2y - 3z = 0$$

One solution is $\boldsymbol{v} = (-5, 4, 1)$.

Step 2. Normalize the vector.

$$\tilde{\boldsymbol{v}} = \frac{\boldsymbol{v}}{||\boldsymbol{v}||} = \frac{1}{\sqrt{42}}(-5, 4, 1)$$

which makes Q orthogonal.

3. **Gram-Schmidt** Consider the following vectors

$$\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \boldsymbol{a}_2 = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, \quad \boldsymbol{a}_3 = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}.$$

(a) Use the Gram-Schmidt process to find an orthonorml basis for the space spanned by $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$.

This is an example on textbook (p238). Let $\boldsymbol{A} = \boldsymbol{a}_1$.

Step 1.

$$\boldsymbol{B} = \boldsymbol{a}_2 - \frac{\boldsymbol{A}^T \boldsymbol{a}_2}{\boldsymbol{A}^T \boldsymbol{A}} \boldsymbol{A} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

Step 2.

$$\boldsymbol{C} = \boldsymbol{a}_3 - \frac{\boldsymbol{A}^T \boldsymbol{a}_3}{\boldsymbol{A}^T \boldsymbol{A}} \boldsymbol{A} - \frac{\boldsymbol{B}^T \boldsymbol{a}_3}{\boldsymbol{B}^T \boldsymbol{B}} \boldsymbol{B} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The lengths of $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ are $\sqrt{2}$ and $\sqrt{6}$ and $\sqrt{3}$. Divide by those lengths, for an orthonormal basis:

$$\boldsymbol{q}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \boldsymbol{q}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \boldsymbol{q}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(b) Write your result in the form $A = QR$.

Columns of Q are $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$. Entries of R are given by the formula $R_{ij} = \boldsymbol{q}_i^T \boldsymbol{a}_j$ when $i \leq j$ and $R_{ij} = 0$ when $i > j$. Thus

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & -3 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{18} \\ 0 & \sqrt{6} & -\sqrt{6} \\ 0 & 0 & \sqrt{3} \end{bmatrix} = QR$$