Sec 3.5, 4.1:

1. Consider the matrix
$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & 4 \\ -1 & -2 & 2 & 1 & -2 \\ 2 & 4 & -1 & 4 & 7 \end{bmatrix}.$$

   (a) Find R, the reduced row-echelon form (RREF) of A.
   We reduce the matrix R as follows:
$$\begin{bmatrix} 1 & 2 & 0 & 3 & 4 \\ -1 & -2 & 2 & 1 & -2 \\ 2 & 4 & -1 & 4 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 2 & 4 & 2 \\ 2 & 4 & -1 & 4 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 2 & 4 & 2 \\ 0 & 0 & -1 & -2 & -1 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 2 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 3 & 4 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   (b) What are the pivot columns of A? What are the free columns of A?
   The pivot columns of A are the first and third columns:
$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$$
   The free columns of A are the second, fourth, and fifth columns:
$$\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix}$$

   (c) Find a basis and the dimension for Col$A$.

   R has 2 pivots, so that A has rank 2. We can therefore choose 2 linearly independent columns of A to get a basis of Col$A$:
$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \right\}$$

(d) Find a basis and the dimension for Nul$A$.

The dimension of the null space is the number of free columns. Since row operations preserve the null space, we seek to find the null space of R. As usual, this is the span of the special solutions. In

$$R \begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix} = 0$$

If we set $x_2 = 1, x_4 = 0, x_5 = 0$, then we must have $x_1 = -2, x_4 = 0, x_5 = 0$. This gives

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Similarly, setting $x_4 = 1, x_2 = 0, x_5 = 0$, we get $x_1 = -3$ and $x_3 = -2$, so that

$$\begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

is a special solution. Lastly, setting $x_5 = 1$, $x_2 = 0$, and $x_4 = 0$, we get

$$\begin{bmatrix} -4 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

as a special solution. So a basis is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(e) Find a basis and the dimension for Row$A$.

$\dim(\text{Row}(A)) = \dim(\text{Col}(A)) = 2$. A basis is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 2 \\ 1 \\ -2 \end{bmatrix} \right\} \quad \text{or} \quad \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

(f) Find a basis and the dimension for $\text{Nul}A^T$.

$\dim(\text{Nul}(A^T)) = 3 - 2 = 1$. We can compute the left nullspace by solving

$$A^T\boldsymbol{x} = \boldsymbol{0} \implies \begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 0 & 2 & -1 \\ 3 & 1 & 4 \\ 4 & -2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus, a basis for $\text{Nul}\, A^T$ is given by

$$\left\{ \begin{bmatrix} -3/2 \\ 1/2 \\ 1 \end{bmatrix} \right\}.$$

2. Why is there no matrix whose row space and nullspace both contain $(1, 1, 1)$?
The row space is the orthogonal complement space of the null space, as the row space is the column space of the transpose. Therefore each element of the row space is perpendicular to each element of the null space. An element v in both of these spaces would have $v \cdot v = 0$. But $(1, 1, 1) \cdot (1, 1, 1) = 3 \neq 0$. So $(1, 1, 1)$ cannot be contained in the intersection of the nullspace and column space.

3. Find the rank of A and write the matrix as $A = uv^T$:
   (a) $A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 6 \end{bmatrix}$.

   We can take u to be the basis of $\mathrm{Col}(A)$ and v to be the basis of $\mathrm{Col}(A^T)$.

$$u = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, v = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}$$

   This gives

$$uv^T = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 6 \end{bmatrix}$$

   (b) $A = \begin{bmatrix} 2 & -2 \\ 6 & -6 \end{bmatrix}$

$$u = \begin{bmatrix} 2 \\ 6 \end{bmatrix}, v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

   This gives

$$uv^T = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 6 & -6 \end{bmatrix}$$

Sec 4.2:

1. If $A\boldsymbol{x} = \boldsymbol{b}$ has a solution and $A^T\boldsymbol{y} = \mathbf{0}$, is $(\boldsymbol{y}^T\boldsymbol{x} = 0)$ or $(\boldsymbol{y}^T\boldsymbol{b} = 0)$? Why?

$$A^T\boldsymbol{y} = \mathbf{0} \implies \left(A^T\boldsymbol{y}\right)^T = \mathbf{0}^T \implies \boldsymbol{y}^T A = \mathbf{0}^T \implies \boldsymbol{y}^T A\boldsymbol{x} = \mathbf{0}^T\boldsymbol{x} = 0 \implies \boldsymbol{y}^T\boldsymbol{b} = 0$$

2. If $A^T A\boldsymbol{x} = \mathbf{0}$ then $A\boldsymbol{x} = \mathbf{0}$. Why? *Hint:* The vector $A\boldsymbol{x}$ is in which of two of the four subspaces?

   See lecture 12 notes.

3. Suppose A is a symmetric matrix ($A^T = A$). Why is its column space perpendicular to its nullspace?

   We have
$$\mathrm{Col}(A)^\perp = \mathrm{Col}(A^T)^\perp = \mathrm{Nul}(A^T) = \mathrm{Nul}(A)$$

4. Find the projection vector, $\boldsymbol{p}$, that results from projecting $\boldsymbol{b}$ onto the column space of A:
$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \boldsymbol{b} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$$

   We have $\boldsymbol{p} = A\hat{\boldsymbol{x}}$, where $A^T A\hat{\boldsymbol{x}} = A^T\boldsymbol{b}$. Then,
$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} \implies \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}.$$

   Therefore
$$\boldsymbol{p} = A\hat{\boldsymbol{x}} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} = \boldsymbol{b}.$$

   Thus, $\boldsymbol{b}$ is contained in the column space of A.