Sec 3.3:

1. Consider
$$A = \begin{bmatrix} 1 & 0 & 3 & -4 \\ -2 & 1 & -6 & 6 \\ 1 & 0 & 2 & -1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$
Find the complete solution to $A\boldsymbol{x} = \boldsymbol{b}$.

The augmented matrix matrix
$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & -4 & 1 \\ -2 & 1 & -6 & 6 & 3 \\ 1 & 0 & 2 & -1 & 2 \end{array}\right]$$
becomes
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 5 & 4 \\ 0 & 1 & 0 & -2 & 5 \\ 0 & 0 & 1 & -3 & -1 \end{array}\right] (*)$$
The fourth column is free, and hence x_4 is a free variable. Let $x_4 = 0$. Then, one particular solution is $\boldsymbol{x_p} = \begin{bmatrix} 4 \\ 5 \\ -1 \\ 0 \end{bmatrix}$.

Now, let the last column of (*) equal to 0. Then, we have
$$x_1 = -5x_4$$
$$x_2 = 2x_4$$
$$x_3 = 3x_4.$$
Then, the special solution (nullspace solution) is $\boldsymbol{x_n} = x_4 \begin{bmatrix} -5 \\ 2 \\ 3 \\ 1 \end{bmatrix}$.

The complete solution is $\boldsymbol{x} = \boldsymbol{x_p} + \boldsymbol{x_n}$.

2. Find the complete solution of the system

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix}.$$

The augmented matrix matrix

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 1 & 3 & 1 & 6 & 7 \end{array}\right]$$

becomes

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] (*)$$

One particular solution is obtained by setting the free variables $x_2 = x_4 = 0$, and hence

$$\boldsymbol{x_p} = \begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \end{bmatrix}$$

Now, let the last column of (*) equal to 0. Then, we have

$$x_1 = -3x_2 + -2x_4$$

$$x_3 = -4x_4$$

Then, the special solution is $\boldsymbol{x_n} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -4 \\ 1 \end{bmatrix}$

The complete solution is $\boldsymbol{x} = \boldsymbol{x_p} + \boldsymbol{x_n}$.

3. Under what condition on b_1, b_2, b_3 is the following system solvable?

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= b_1 \\ 2x_1 + 5x_2 - 4x_3 &= b_2 \\ 4x_1 + 9x_2 - 8x_3 &= b_3 \end{aligned}$$

Generally speaking, $\boldsymbol{b}$ must be in the column space of the corresponding matrix. After elimination, we get

$$\left[\begin{array}{ccr|l} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - 2b_1 - b_2 \end{array}\right]$$

The system is solvable only when $b_3 - 2b_1 - b_2 = 0$.

4. Choose the number q so that (if possible) the ranks of A and B are (i) 1, (ii) 2, (iii) 3:

$$A = \begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 1 & 3 \\ q & 2 & q \end{bmatrix}.$$

Observe that the second column of A is $\frac{2}{3}$ of the first column.
So, A cannot have rank 3. When $q = 3$, A has rank 1. When $q \neq 3$, A has rank 2.

In B, the first column and the third column are the same no matter what q is.
So, B cannot have rank 3. When $q = 6$, B has rank 1. When $q \neq 6$, B has rank 2.

Sec 3.4:

1. Consider the following subspaces Y, W, and V.
   (a) Describe each subspace (line, plane, etc).
   (b) Determine which of these form a **basis** for the subspace? Why or why not?
   (c) What is the **dimension** of each subspace?

i.
$$Y = \text{span}\left\{ \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Y is plane in $\mathbb{R}^3$ passing through origin.
A basis for Y is given by:

$$\beta_Y = \left\{ \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$\dim(Y) = 2$.

ii.
$$W = \text{span}\left\{ \begin{bmatrix} 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \end{bmatrix} \right\}$$

W spans $\mathbb{R}^2$.
The set of vectors is not a basis for $\mathbb{R}^2$ as presented, because the vectors are not linearly independent — any two of these vectors constitutes a basis for $\mathbb{R}^2$, e.g.,

$$\beta_{\mathbb{R}^2} = \left\{ \begin{bmatrix} 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$$

$\dim(W) = 2$.

iii.
$$V = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 1 \\ 12 \end{bmatrix} \right\}$$

Note that , $\begin{bmatrix} 6 \\ 0 \\ 1 \\ 12 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ -1 \\ 2 \end{bmatrix}$.

Thus, the set of vectors forms a plane in $\mathbb{R}^4$.
A basis for V is given by:

$$\beta_V = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -1 \\ 2 \end{bmatrix} \right\}$$

$\dim(V) = 2$.

2. Consider the following matrix-vector system

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 4 & 8 \\ 4 & 8 & 6 & 8 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 4 \\ 2 \\ 10 \end{bmatrix}$$

   (a) Find a **basis** for the column space of the matrix A. What is the rank of A?
   We do elimination, and we get the REF of A to be

$$\begin{bmatrix} \boxed{1} & 2 & 1 & 0 \\ 0 & 0 & \boxed{2} & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\beta_{\text{Col}A} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} \right\}$$

   The rank of A is 2.

   (b) Find a **basis** for the row space of the matrix A.
   A basis for the row space is given by the nonzero rows in the REF of A.

$$\beta \text{Row} A = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 8 \end{bmatrix} \right\}$$

   (c) Find a **basis** for the nullspace of the matrix A.
   $\boldsymbol{x}$ is in the nullspace of A if $A\boldsymbol{x} = \mathbf{0}$.

$$\text{Nul}A = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}$$

   And therefore, a basis for Nul$A$ is given by:

$$\beta_{\text{Nul}A} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}$$

(d) Find the complete solution $\boldsymbol{x} = \boldsymbol{x_p} + \boldsymbol{x_n}$.

We solve the system

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 4 & 8 \\ 4 & 8 & 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 10 \end{bmatrix}$$

In augmented form, this gives

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 4 \\ 2 & 4 & 4 & 8 & 2 \\ 4 & 8 & 6 & 8 & 10 \end{array}\right],$$

which after elimination gives

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 4 \\ 0 & 0 & 2 & 8 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Note that x_1 and x_3 are pivot variables, while x_2 and x_4 are free variables. Thus, we obtain that $x_1 = 7 - 2x_2 + 4x_4$ and $x_3 = -3 - 4x_4$, and hence

$$\boldsymbol{x} = \boldsymbol{x_p} + \boldsymbol{x_n} = \begin{bmatrix} 7 \\ 0 \\ -3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 0 \\ -4 \\ 1 \end{bmatrix}.$$

(e) Choose a different $\boldsymbol{x_p}$ from the one used in Part (a). Show that $\boldsymbol{x} = \boldsymbol{x_p} + \boldsymbol{x_n}$, with the new $\boldsymbol{x_p}$, still solves the system.

For example, let $x_2 = 1$ and $x_4 = 0$. Then, $\begin{bmatrix} 5 \\ 1 \\ -3 \\ 0 \end{bmatrix}$ is another $\boldsymbol{x_p}$.

3. Consider the following matrix
$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{bmatrix}$$

(a) What's in the nullspace of A?
Since the columns of A are linearly independent, then Nul$A = \{\mathbf{0}\}$.

(b) For any $\boldsymbol{b}$, how many solutions do you expect $A\boldsymbol{x} = \boldsymbol{b}$ to have?
Either exactly one solution or no solution.

(c) What is the condition on $\boldsymbol{b} = [b_1,\ b_2,\ b_3]$ such that $A\boldsymbol{x} = \boldsymbol{b}$ is solvable?
Consider the system

$$\left[\begin{array}{rr|l} 1 & 1 & b_1 \\ 1 & 2 & b_2 \\ -2 & -3 & b_3 \end{array}\right] \rightarrow \left[\begin{array}{rr|l} 1 & 1 & b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & -1 & b_3 + 2b_1 \end{array}\right] \rightarrow \left[\begin{array}{rr|l} 1 & 1 & b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_3 + b_2 + b_1 \end{array}\right]$$

Hence, the system is solvable when $b_1 + b_2 + b_3 = 0$.

4. Consider the following matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

(a) If $\boldsymbol{b} = \begin{bmatrix} 3 \\ 6 \\ \beta \end{bmatrix}$, for what values of the scalar β will $A\boldsymbol{x} = \boldsymbol{b}$ have a solution?

In augmented form, the system becomes

$$\left[\begin{array}{ccccc|c} 0 & 1 & 2 & 3 & 4 & 3 \\ 0 & 1 & 2 & 4 & 6 & 6 \\ 0 & 0 & 0 & 1 & 2 & \beta \end{array}\right].$$

We do elimination, and we get

$$\left[\begin{array}{ccccc|c} 0 & 1 & 2 & 3 & 4 & 3 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 & \beta \end{array}\right].$$

Hence, for the system to be solvable, β must be 3.

(b) For the β from part (a), find the **complete** solution to $A\boldsymbol{x} = \boldsymbol{b}$.

Let $\beta = 3$ and solve

$$\left[\begin{array}{ccccc|c} 0 & 1 & 2 & 3 & 4 & 3 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{array}\right] \to \left[\begin{array}{ccccc|c} 0 & \boxed{1} & 2 & 3 & 4 & 3 \\ 0 & 0 & 0 & \boxed{1} & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Hence, x_1, x_3, and x_5 are free variables.

Row 2: $\boxed{x_4 = 3 - 2x_5}$

Row 1: $x_2 = 3 - 2x_3 - 3x_4 - 4x_5 = 3 - 2x_3 - 3(3 - 2x_5) - 4x_5 \implies \boxed{x_2 = -6 - 2x_3 + 2x_5}$

Thus, the complete solution is given by:

$$\boldsymbol{x} = \boldsymbol{x_p} + \boldsymbol{x_n} = \begin{bmatrix} 0 \\ -6 \\ 0 \\ 3 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$