Sec 3.1:

1. Which of the following subsets of $\mathbb{R}^3$ are subspaces? Justify your answer.

   (a) The plane of vectors (b_1, b_2, b_3) that satisfy $b_1 = 0$.

   Let $B = \{\boldsymbol{b} \in \mathbb{R}^3 : b_1 = 0\}$

   $\mathbf{0} \in B$ (evident)

   Let $\boldsymbol{u} = \begin{pmatrix} 0 \\ u_2 \\ u_3 \end{pmatrix}, \boldsymbol{v} = \begin{pmatrix} 0 \\ v_2 \\ v_3 \end{pmatrix}$, and $a, b \in \mathbb{R}$

   $$a\boldsymbol{u} + b\boldsymbol{v} = \begin{pmatrix} 0 \\ au_1 \\ au_2 \end{pmatrix} + \begin{pmatrix} 0 \\ bv_2 \\ bv_3 \end{pmatrix} = \begin{pmatrix} 0 \\ au_1 + bv_1 \\ au_2 + bu_2 \end{pmatrix} \in B$$

   $\therefore$ B is a subspace.

   (b) The plane of vectors (b_1, b_2, b_3) that satisfy $b_1 = 1$.

   Let $V = \{\boldsymbol{v} \in \mathbb{R}^3 : v_1 = 1\}$

   (1) $\begin{pmatrix} 1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} 1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 2 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix} \notin V$. So, V is not closed under addition.

   (2) $c \begin{pmatrix} 1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} c \\ cv_2 \\ cv_3 \end{pmatrix} \notin V$. So, V is not closed under scalar multiplication.

   (3) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \notin V$.

   Any one of these is sufficient to show that V is not a subspace.

(c) All combinations of two given vectors $(1, 1, 0)$ and $(2, 0, 1)$.

First, observe that $\mathbf{0}$ is a linear combination of any two vectors, in particular the given ones.

$$\alpha\left(a\begin{pmatrix}1\\1\\0\end{pmatrix}+b\begin{pmatrix}2\\0\\1\end{pmatrix}\right)+\beta\left(c\begin{pmatrix}1\\1\\0\end{pmatrix}+d\begin{pmatrix}2\\0\\1\end{pmatrix}\right)$$

$$=\alpha a\begin{pmatrix}1\\1\\0\end{pmatrix}+\alpha b\begin{pmatrix}2\\0\\1\end{pmatrix}+\beta c\begin{pmatrix}1\\1\\0\end{pmatrix}+\beta d\begin{pmatrix}2\\0\\1\end{pmatrix}$$

$$=(\alpha a+\beta c)\begin{pmatrix}1\\1\\0\end{pmatrix}+(\alpha b+\beta d)\begin{pmatrix}2\\0\\1\end{pmatrix}$$

$\therefore$ It is a subspace.

(d) The plane of vectors (b_1, b_2, b_3) that satisfy $b_3 - b_2 + 3b_1 = 0$.

Let $U = \{u \in \mathbb{R}^3 : u_3 - u_2 + 3u_1 = 0\}$. Observe that $\mathbf{0} \in U$ since $0 - 0 + 3(0) = 0$. Let $\boldsymbol{u}$, $\boldsymbol{v} \in U$. Then

$$a\boldsymbol{u} + b\boldsymbol{v} = \begin{pmatrix}au_1+bv_1\\au_2+bv_2\\au_3+bv_3\end{pmatrix}$$

$$au_3 + bv_3 - (au_2 + bv_2) + 3(au_1 + bv_1) = (au_3 - au_2 + 3au_1) + (bv_3 - bv_2 + 3bv_1)$$

$$= a(u_3 - u_2 + 3u_1) + b(v_3 - v_2 + 3v_1) = 0a + 0b = 0$$

$\implies a\boldsymbol{u} + b\boldsymbol{v} \in U$.
$\therefore$ U is a subspace.

2. The set $W \subseteq \mathbb{R}^3$ is the set of all vectors $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ whose coordinates satisfy

$$x_1 - x_2 + 2x_3 = 0$$

$$3x_2 - x_3 = 0$$

Determine if W is a subspace of $\mathbb{R}^3$.

First note that the zero vector in $\mathbb{R}^3$ is in W since its components verify both equations above. Solving the system we have:

$$3x_2 - x_3 = 0 \implies \boxed{x_3 = 3x_2}$$

$$\implies x_1 - x_2 + 2(3x_2) = 0 \implies \boxed{x_1 = -5x_2}$$

So, $W = \{\boldsymbol{x} \in \mathbb{R}^3 : \boldsymbol{x} = t(-5, 1, 3),\ t \in \mathbb{R}\}$

$$a\left(t_1 \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}\right) + b\left(t_2 \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}\right)$$

$$= at_1 \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix} + bt_2 \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}$$

$$= (at_1 + bt_2) \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix} \in W$$

$\therefore W$ is a subspace of $\mathbb{R}^3$

3. (a) Show that the set of nonsingular 2×2 matrices is not a subspace.
Nonsingular $\iff$ Invertible

- $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible and therefore not in the set.
- $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. So the set is not closed under addition.
- $0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. So the set is not closed under scalar multiplication.

Any one of the these is sufficient to show that this set is not a subspace.

(b) Show also that the set of singular 2×2 matrices is not a subspace.
Singular $\iff$ Not invertible

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Which is invertible. So the set is not closed under addition.

Therefore, the set is not a subspace.

4. Describe the column spaces (lines or planes) of these particular matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}$$

The column space is the set of all combinations of the columns.

$\boldsymbol{C}(A) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \right\} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$. This is a line.

$\boldsymbol{C}(B) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \right\}$. This is a plane.

$\boldsymbol{C}(C) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right\}$. This is a line.

Sec 3.2:

1. Consider the following matrix

$$A = \begin{bmatrix} 1 & 0 & 3 & -4 \\ -2 & 1 & -6 & 6 \\ 1 & 0 & 2 & -1 \end{bmatrix}$$

(a) Perform elimination on A until upper triangular matrix appears.

$$A = \begin{bmatrix} 1 & 0 & 3 & -4 \\ -2 & 1 & -6 & 6 \\ 1 & 0 & 2 & -1 \end{bmatrix} \xrightarrow{R_2+2R_1 \ \& \ R_3-R_1} \begin{bmatrix} 1 & 0 & 3 & -4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & -1 & 3 \end{bmatrix}$$

(b) Identify the pivot columns (put a square around the pivots) and the free columns (circle the whole column). What are the free variables? What are the pivot variables?

x_4 is the free variable. x_1, x_2, and x_3 are the pivot variables.

(c) Perform back substitution, writing the pivot variables in terms of the free variables.

$$x_1 + 3x_3 - 4x_4 = 0$$
$$x_2 - 2x_4 = 0$$
$$-x_3 + 3x_4 = 0$$

$\implies x_3 = 3x_4$, $x_2 = 2x_4$, and $x_1 = -9x_4 + 4x_4 = -5x_4$.
So,

$$\boldsymbol{x} = x_4 \begin{pmatrix} -5 \\ 2 \\ 3 \\ 1 \end{pmatrix}$$

(d) Describe the nullspace of A.

Nul(A)=Span$\left\{ \begin{pmatrix} -5 \\ 2 \\ 3 \\ 1 \end{pmatrix} \right\}$.

2. Consider the following matrices

$$A = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix}$$

(a) Compute REF(A) and REF(B). What are the ranks of A and B?

REF(A) : $\begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix} \xrightarrow{R_2-R_1 \text{ \& } R_3-R_1} \begin{bmatrix} 2 & 4 & 6 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & -1 & -2 \end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix} 2 & 4 & 6 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank(A) $= 2$

REF(B): $\begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix} \xrightarrow{R_2-3R_1} \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

Rank(B) $= 2$

(b) Find Col$A$, and describe it geometrically.

$\text{Col}A = \text{Span}\left\{ \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix} \right\}$. A plane in $\mathbb{R}^3$.

(c) Find Col$B$, and describe it geometrically.

$\text{Col}B = \text{Span}\left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 8 \end{pmatrix} \right\}$. A plane in $\mathbb{R}^2$, i.e., all of $\mathbb{R}^2$.

(d) Compute RREF(A) and RREF(B).

RREF(A): $\begin{bmatrix}1&2&3&2\\0&1&1&2\\0&0&0&0\end{bmatrix} \xrightarrow{R_1-2R_2} \begin{bmatrix}1&0&1&-2\\0&1&1&2\\0&0&0&0\end{bmatrix}$

RREF(B): $\begin{bmatrix}1&2&2&4\\0&1&0&2\end{bmatrix} \xrightarrow{R_1-2R_2} \begin{bmatrix}1&0&2&0\\0&1&0&2\end{bmatrix}$

(e) Find Nul$A$, and describe it geometrically.

$$\begin{bmatrix}1&0&1&-2\\0&1&1&2\\0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \mathbf{0} \implies \begin{aligned}x_1+x_3-2x_4&=0\\x_2+x_3+2x_4&=0\end{aligned} \implies \begin{aligned}x_1&=-x_3+2x_4\\x_2&=-x_3-2x_4\\x_3&=\text{free}\\x_4&=\text{free}\end{aligned}$$

$$\boldsymbol{x} = \begin{pmatrix}-x_3+2x_4\\-x_3-2x_4\\x_3\\x_4\end{pmatrix} = x_3\begin{pmatrix}-1\\-1\\1\\0\end{pmatrix} + x_4\begin{pmatrix}2\\-2\\0\\1\end{pmatrix}$$

$$\text{Nul}A = \text{Span}\left\{\begin{pmatrix}-1\\-1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\-2\\0\\1\end{pmatrix}\right\}.$$ A plane in $\mathbb{R}^4$

(f) Find Nul$B$, and describe it geometrically.

$$\begin{bmatrix}1&0&2&0\\0&1&0&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \mathbf{0} \implies \begin{aligned}x_1+2x_3&=0\\x_2+2x_4&=0\end{aligned} \implies \begin{aligned}x_1&=-2x_3\\x_2&=-2x_4\\x_3&=\text{free}\\x_4&=\text{free}\end{aligned}$$

$$\boldsymbol{x} = \begin{pmatrix}-2x_3\\-2x_4\\x_3\\x_4\end{pmatrix} = x_3\begin{pmatrix}-2\\0\\1\\0\end{pmatrix} + x_4\begin{pmatrix}0\\-2\\0\\1\end{pmatrix}$$

$$\text{Nul}B = \text{Span}\left\{\begin{pmatrix}-2\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\-2\\0\\1\end{pmatrix}\right\}.$$ A plane in $\mathbb{R}^4$