Sec 2.5:

1. Suppose A is invertible, and you exchange its first two rows to reach B.

   (a) Is the new matrix B invertible? Why?
   The inverse exists iff elimination produces n pivots (row exchanges are allowed)
   Therefore, $\exists$ elimination matrices s.t. $E_n E_{n-1} \ldots E_2 E_1 P A = U$ where U has none zero diagonal elements.
   $B = P_{12} A \Rightarrow A = P_{12} B \Rightarrow E_n E_{n-1} \ldots E_2 E_1 P P_{12} B = U$
   Thus, elimination of B produces n pivots $\Rightarrow$ B is invertible.

   (b) How would you find B^{-1} from A^{-1}?
   $B^{-1} = (P_{12} A)^{-1} = A^{-1} P_{12}^{-1} = A^{-1} P_{12}^{T} = A^{-1} P_{12}$

2. If the product $M = ABC$ of three matrices is invertible, then A, B, and C are invertible. Find a formula for B^{-1} that involves M^{-1}, A, and C.
   $M = ABC \Rightarrow A^{-1} M = BC \Rightarrow A^{-1} M C^{-1} = B \Rightarrow B^{-1} = (A^{-1} M C^{-1})^{-1} \Rightarrow B = C M^{-1} A$

3. For which three numbers c is this matrix not invertible, and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

$$A = \begin{pmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{pmatrix} \to \begin{pmatrix} 2-c & 0 & 0 \\ c & c & c \\ 8 & 7 & c \end{pmatrix} \xrightarrow{c \neq 0} \begin{pmatrix} 2-c & 0 & 0 \\ 1 & 1 & 1 \\ 8 & 7 & c \end{pmatrix} \xrightarrow{c \neq 2} \begin{pmatrix} 2-c & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 7 & c \end{pmatrix}$$

$$\to \begin{pmatrix} 2-c & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & c-7 \end{pmatrix}$$

Therefore, we need $c \neq 0, 2, 7$ to make A invertible

4. Find the inverse using Gauss-Jordan elimination:

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 1 \\ -2 & 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$[A|I] \to [I|A^{-1}]$

$$[A|I] = \begin{pmatrix} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 1 & 0 \\ -2 & 0 & 2 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 1 & 0 \\ 0 & 0 & \frac{2}{3} & 1 & -\frac{1}{3} & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{1}{2} & \frac{3}{2} \end{pmatrix} \to \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{1}{2} & \frac{3}{2} \end{pmatrix} \to \begin{pmatrix} 1 & \frac{1}{2} & 0 & \frac{5}{4} & -\frac{1}{4} & \frac{3}{4} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 & \frac{3}{2} & -\frac{1}{2} & 1 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{1}{2} & \frac{3}{2} \end{pmatrix} \Rightarrow A^{-1} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & 1 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

$$[B|I] = \begin{pmatrix} 2 & 1 & -1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{3}{5} & \frac{1}{5} & \frac{2}{5} & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{3}{5} & \frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 0 & \frac{7}{5} & \frac{1}{5} & \frac{2}{5} & 1 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{3}{5} & \frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 0 & 1 & \frac{1}{7} & \frac{2}{7} & \frac{5}{7} \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & \frac{2}{7} & \frac{4}{7} & \frac{3}{7} \\ 0 & 0 & 1 & \frac{1}{7} & \frac{2}{7} & \frac{5}{7} \end{pmatrix} \to \begin{pmatrix} 1 & \frac{1}{2} & 0 & \frac{4}{7} & \frac{1}{7} & \frac{5}{14} \\ 0 & 1 & 0 & \frac{2}{7} & \frac{4}{7} & \frac{3}{7} \\ 0 & 0 & 1 & \frac{1}{7} & \frac{2}{7} & \frac{5}{7} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & \frac{3}{7} & -\frac{1}{7} & \frac{1}{7} \\ 0 & 1 & 0 & \frac{2}{7} & \frac{4}{7} & \frac{3}{7} \\ 0 & 0 & 1 & \frac{1}{7} & \frac{2}{7} & \frac{5}{7} \end{pmatrix}$$

$$\Rightarrow B^{-1} = \begin{pmatrix} \frac{3}{7} & -\frac{1}{7} & \frac{1}{7} \\ \frac{2}{7} & \frac{4}{7} & \frac{3}{7} \\ \frac{1}{7} & \frac{2}{7} & \frac{5}{7} \end{pmatrix}$$

Sec 2.6, 2.7:

1. **LU Factorization.** Complete the following steps to find the LU factorization of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 0 & 4 & 1 \end{bmatrix}$$

(a) Find the two elimination matrices, E_{21}, E_{32} that will put A into upper-triangular form.

$$E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

$$E_{32}E_{21}A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -5 \end{pmatrix} = U$$

(b) Find E_{21}^{-1} and E_{32}^{-1} and multiply them in the correct order to find L such that $A = LU$.

$$L = (E_{32}E_{21})^{-1} = E_{21}^{-1}E_{32}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

(c) Find the diagonal matrix D such that $A = LDU$, where both L and U have ones on their diagonals.

$$A = LDU = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -5 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

(d) Check that your factorization works! Multiply your L, D, and U matrices to recover A.

repeat the matrix multiplication.

2. Consider the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix}$$

(a) Compute the LU-factorization to find matrices L and U such that $A = LU$.

$$E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

$$E_{32}E_{21}A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & -5 \end{pmatrix} = U$$

$$L = (E_{32}E_{21})^{-1} = E_{21}^{-1}E_{32}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & -5 \end{pmatrix}$$

(b) Switch rows 2 and 3 of A to find S. Show that this new matrix S is symmetric.

$$S = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

$S^T = S \Rightarrow$ S is symmetric

(c) Compute the factorization $S = LDL^T$.

$E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$, $E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix}$

$E_{32}E_{31}S = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -\frac{5}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -\frac{5}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} = DL^T$

check $L = (E_{32}E_{31})^{-1} = E_{31}^{-1}E_{32}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -\frac{1}{2} & 1 \end{pmatrix}$

$S = LDL^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -\frac{5}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$

3. What three matrices E_{21}, E_{12} and D reduce $A = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$ to the identity matrix (i.e., $DE_{12}E_{21}A = I$)? Multiply these matrices together to find A^{-1}.

   $E_{21} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

   $DE_{12}E_{21}A = I$

   $A^{-1} = DE_{12}E_{21} = \begin{pmatrix} 3 & -1 \\ -1 & \frac{1}{2} \end{pmatrix}$

4. Solve the system $A\boldsymbol{x} = \boldsymbol{b}$ using the LU factorization of A, where

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 8 & 9 \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}.$$

(a) First factor A into LU, and obtain the system $LU\boldsymbol{x} = \boldsymbol{b}$

$$E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}, E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{5} & 1 \end{pmatrix}$$

$$E_{32}E_{31}A = \begin{pmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & \frac{7}{5} \end{pmatrix} = U$$

$$L = (E_{32}E_{31})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -\frac{1}{5} & 1 \end{pmatrix}$$

$$A = LU = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -\frac{1}{5} & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & \frac{7}{5} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -\frac{1}{5} & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & \frac{7}{5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix}$$

(b) Use your result in (a) to compute $L\boldsymbol{y} = \boldsymbol{b}$. What is $\boldsymbol{y}$?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -\frac{1}{5} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -\frac{3}{5} \end{pmatrix}$$

(c) Use your result in (b) to find $\boldsymbol{x}$.

$$\begin{pmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & \frac{7}{5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -\frac{3}{5} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{7} \\ 1 \\ -\frac{3}{7} \end{pmatrix}$$

(d) How could you factor A into a product UL, upper triangular times lower triangular? Would they be the same factors as in $A = LU$?

use elimination matrices to make A into lower triangular matrix and then find U = (all elimination matrices)$^{-1}$. They would probably not be the same factor.