Sec 2.1, 2.2:

1. Elimination: three equations, three unknowns. Consider the following system of equations

$$x + 2y - z = 1 \tag{1a}$$
$$2x - y + z = 3 \tag{1b}$$
$$3x + y - 2z = 4 \tag{1c}$$

   (a) Eliminate x.

      i. Find the pivot in Eq. (1a) and multipliers from Eqs. (1b) and (1c) that will eliminate the x terms in Eqs. (1b) and (1c).
      Pivot: 1

      Multiplier from Eq. (1b): 2
      Multiplier from Eq. (1c): 3

      ii. Perform the elimination of x from the last two equations.
      $x + 2y - z = 1$ (1)
      $-5y + 3z = 1$ (2′)
      $-5y + z = 1$ (3′)

      iii. Write the resulting 2D system of equations (with just y and z) in which x is eliminated. *Hint:* Leave Eq. (1a) out.
      $-5y + 3z = 1$ (2′)
      $-5y + z = 1$ (3′)

(b) Eliminate y. From answer to (a)iii, find the pivot and multiplier to eliminate y. Perform the elimination and write the final equation that just contains z.
Pivot=-5
$-5y + 3z = 1 \quad (2')$
$-2z = 0 \quad (3'')$

(c) Back substitution. Now, write the original Eq. (1a), the x-eliminated Eq. (1b) (that just contains y and z) and the x,y-eliminated Eq. (1c) (just containing z). You should see a triangular system. Use back substitution to solve the system for z, then y, then x.
$x + 2y - z = 1 \quad (1), \quad x = 7/5$
$-5y + 3z = 1 \quad (2'), \quad y = -1/5$
$-2z = 0 \quad (3''), \quad z = 0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7/5 \\ -1/5 \\ 0 \end{bmatrix}$$

2. Use Gauss-Jordan elimination to solve the following system:

$$2x_1 + 4x_2 - 2x_3 = 2 \tag{2a}$$

$$4x_1 + 9x_2 - 3x_3 = 8 \tag{2b}$$

$$-2x_1 - 3x_2 + 7x_3 = 4 \tag{2c}$$

$2x_1 + 4x_2 - 2x_3 = 2 \quad (1)$
$x_2 + x_3 = 4 \quad (2) - 2 \cdot (1) = (2')$
$x_2 + 5x_3 = 6 \quad (3) + (1) = (3')$

$2x_1 + 4x_2 - 2x_3 = 2 \quad (1)$
$x_2 + x_3 = 4 \quad (2) - 2 \cdot (1) = (2')$
$4x_3 = 2 \quad (3') - (2') = (3'')$

From back substitution, we see that:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -11/2 \\ 7/2 \\ 1/2 \end{bmatrix}$$

3. Are the vectors $\boldsymbol{u} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$, and $\boldsymbol{w} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ linearly independent?

   Hint: Solve $c_1\boldsymbol{u} + c_2\boldsymbol{v} + c_3\boldsymbol{w} = \mathbf{0}$.

$$c_1 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 && (1) \\ 3c_1 + 3c_2 - c_3 &= 0 && (2) \\ c_1 - c_2 + c_3 &= 0 && (3) \end{aligned}$$

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 && (1) \\ -4c_3 &= 0 && (2) - 3(1) \implies c_3 = 0 \\ -2c_2 &= 0 && (3) - (1) \implies c_2 = 0 \end{aligned}$$

The only solution is $c_1 = c_2 = c_3 = 0$, which implies $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent.

4. Use elimination to solve

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \qquad (1) \\ x_1 + 2x_2 + 2x_3 &= 11 \qquad (2) \\ 2x_1 + 3x_2 - 4x_3 &= 3 \qquad (3) \end{aligned}$$

$x_1 + x_2 + x_3 = 6$ (1)
$x_2 + x_3 = 5$ $(2) - (1) = (2')$
$x_2 - 6x_3 = -9$ $(3) - 2(1) = (3')$

$x_1 + x_2 + x_3 = 6$ (1)
$x_2 + x_3 = 5$ $(2')$
$-7x_3 = -14$ $(3') - (2') = (3'')$

From back substitution we have: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

Sec 2.3, 2.4:

1. Write down the 3×3 matrices that produce these steps:

   (a) Subtracts 5 times row 1 from row 2.
   $$\begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   (b) Adds 7 times row 2 to row 3.
   $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}$$

   (c) Exchanges rows 1 and 2, then rows 2 and 3.
   $$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

   (d) Subtracts row 1 from row 2, and then exchanges rows 2 and 3.
   $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

   (e) Exchanges rows 2 and 3, and then subtracts row 1 from row 3.
   $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

2. Elimination matrices: Consider the following system of equations:

$$x + 2y - z = 1$$
$$2x - y + z = 3$$
$$3x + y - 2z = 4$$

We are going to go through the process of solving a linear system using elimination matrices.

(a) Write the system of equations as a matrix vector system: $A\boldsymbol{x} = \boldsymbol{b}$.

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

row 1 column 1 in the matrix to the left is the pivot.

(b) Identify the pivot in the first column (circle it in your matrix A). Write the two elimination matrices E_{21} and E_{31} that, when applied to A, will give zeros in the necessary positions.

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

(c) Perform the multiplication $E_{31}E_{21}A$. Do you see zeros below the pivot?

$$E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 0 & -5 & 1 \end{bmatrix}$$

Row 2 column 2 is the new pivot.

(d) Now move on to the second column. Identify the new pivot and circle it in your new matrix from Part (c) [it might be useful to re-write that matrix here]. *Hint:* recall that we are looking to write the system as an upper-triangular system. Therefore, we are looking to get a zero under all of the diagonal entries of A.
Please see above.

(e) Write the elimination matrix E_{32} that, when applied to the matrix from Part (c), will give a zero in the necessary position. Multiply this elimination matrix with your matrix from Part (c). Do you see an upper-triangular matrix?

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$E_{32}E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 0 & -5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 0 & 0 & -2 \end{bmatrix}$$

(f) Multiply out the three elimination matrices from Parts (c) and (e): $E = E_{32}E_{31}E_{21}$.
Using matrix multiplication, we see that:

$$E = E_{32}E_{31}E_{21}$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

(g) Now that you have one matrix, E, that describes all elimination operations performed on A, we can solve the system by applying this matrix to both sides of the equation: $EA\boldsymbol{x} = E\boldsymbol{b}$. Write out this new matrix-vector system.

$EA\boldsymbol{x} = E\boldsymbol{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

(h) Find the solution $\boldsymbol{x}$ by performing back substitution on the system from Part (g). *Hint:* Convert back to a system of equations.

$x_1 + 2x_2 - x_3 = 1 \implies x_1 = 7/5$

$-5x_2 + 3x_3 = 1 \implies x_2 = -1/5$

$-2x_3 = 1 \implies x_3 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7/5 \\ -1/5 \\ 0 \end{bmatrix}$$

(i) Check your answer by substituting the answer back into the original system in Part (a).

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} 7/5 \\ -1/5 \\ 0 \end{bmatrix} = \begin{bmatrix} 7/5 - 2/5 \\ 14/5 + 1/5 \\ 21/5 - 1/5 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \boldsymbol{b}$$

3. Which three elimination matrices put A into triangular form U?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix}$$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

4. Which elimination matrices put A into triangular form U?

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_{21}A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2/3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_{32}E_{21}A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3/4 & 1 \end{bmatrix}$$

$$E_{43}E_{32}E_{21}A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}$$