Sec 1.1, 1.2:

1. Consider the following two vectors:

$$\vec{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Draw and label the following vectors on the axis provided:

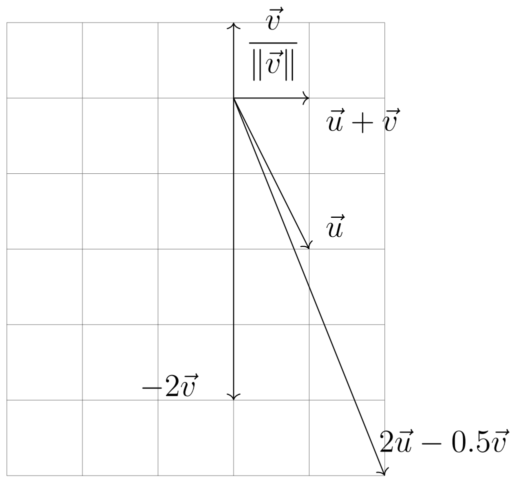
(a) $\vec{u}$

(b) $-2\vec{v}$

(c) $\vec{u} + \vec{v}$

(d) $2\vec{u} - 0.5\vec{v}$

(e) The *unit* vector, $\frac{\vec{v}}{||\vec{v}||}$



What are the lengths of $\vec{u}$ and $\vec{v}$?

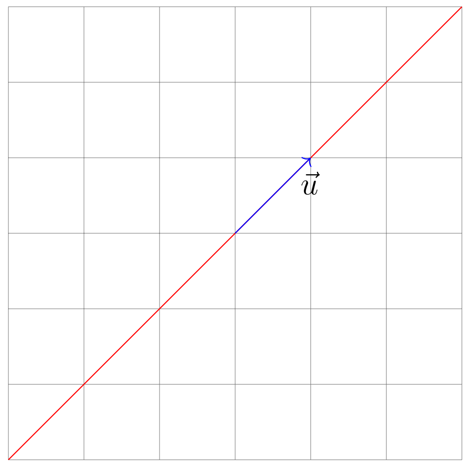
$\|\vec{u}\| = \sqrt{1 + (-2)^2} = \sqrt{5}$, $\|\vec{v}\| = \sqrt{2^2} = 2$

2. True or False? _____ Any list of 7 real numbers is a vector in $\mathbb{R}^7$.

True

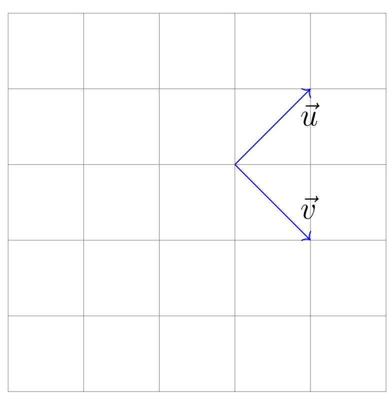
3. Linear Combinations:

(a) Let $\vec{u}$ be the vector $\begin{bmatrix}1\\1\end{bmatrix}$ in $\mathbb{R}^2$. What does the set of all possible combinations, $c\vec{u}$, $c \in \mathbb{R}$ look like? This is called the **span** of $\vec{u}$, or $\text{span}\{\vec{u}\}$.
Hint: Try some values for c such as $c = 2, c = -4$, etc.



Span is indicated in red.

(b) Let $\vec{v}$ be another vector $\begin{bmatrix}1\\-1\end{bmatrix}$ in $\mathbb{R}^2$. What does the set of all combinations, $c_1\vec{u} + c_2\vec{v}$, where $c_1, c_2 \in \mathbb{R}$, look like? This is the $\text{span}\{\vec{u}, \vec{v}\}$.
Hint: Try $c_1 = 0$ and $c_2 = 0$ first, then some combinations of the two.



$\vec{u}$ and $\vec{v}$ are not parallel, and thus span all of $\mathbb{R}^2$.

(c) What about if $\vec{v} = \begin{bmatrix}-1\\-1\end{bmatrix}$? What does the set of all combinations, $c_1\vec{u}+c_2\vec{v}$, where $c_1, c_2 \in \mathbb{R}$, look like?
$\vec{u}$ and $\vec{v}$ are parallel, so in this case the span is a line in $\mathbb{R}^2$.

(d) Let $\vec{u}, \vec{v}, \vec{w}$ be arbitrary vectors in $\mathbb{R}^3$. What does the set of all combinations, $c_1\vec{u} + c_2\vec{v} + c_3\vec{w}$, where $c_1, c_2, c_3 \in \mathbb{R}$, look like? This is the span$\{\vec{u}, \vec{v}, \vec{w}\}$.

all parallel: a line in $\mathbb{R}^3$
2 vectors parallel: a plane in in $\mathbb{R}^3$
none parallel (and all three vectors not in the same plane): all of $\mathbb{R}^3$

4. Given the set of vectors:

$$\vec{u} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Compute the following:

(a) The unit vector in the direction of $\vec{u}$.

$$\|\vec{u}\| = \sqrt{1^2 + 3^2 + 4^2} = \sqrt{26}\ , \ \frac{\vec{u}}{\|\vec{u}\|} = \begin{bmatrix} \frac{1}{\sqrt{26}} \\ \frac{3}{\sqrt{26}} \\ \frac{4}{\sqrt{26}} \end{bmatrix}$$

(b) The unit vector in the direction of $\vec{v}$.

$$\|\vec{v}\| = \sqrt{0^2 + 1^2 + 2^2} = \sqrt{5}\ , \ \frac{\vec{v}}{\|\vec{v}\|} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

(c) The dot product, $\vec{u} \cdot \vec{v}$.

$$\vec{u} \cdot \vec{v} = 1(0) + 3(1) + 4(2) = 11$$

(d) The cosine of the angle between $\vec{u}$ and $\vec{v}$.

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{11}{\sqrt{26}\sqrt{5}} = \frac{11}{\sqrt{130}}$$

5. Cosine Formula:

(a) What does the set of all unit vectors in $\mathbb{R}^2$ look like?

A filled circle of radius 1.

(b) Consider the unit vector $\vec{U} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$ and the vector along the x-axis, $\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. What is $\vec{U} \cdot \vec{i}$?

$\vec{U} \cdot \vec{i} = \cos\alpha$

(c) Now instead consider a generic unit vector $\vec{u} = \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix}$. What is $\vec{u} \cdot \vec{U}$? Consider the identity $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$, let $\theta = \alpha - \beta$, and simplify your answer.

$\vec{u} \cdot \vec{U} = \cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta) = \cos\theta$

(d) Suppose we have two vectors, u and v that are not unit vectors. Make these vectors unit by dividing by their lengths. Then, apply the formula in (c) to verify the Cosine formula: $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$. What is θ here and how does it relate to the α and β in (c)? *Hint:* It might be useful to draw out the vectors.

$\|\vec{u}\| = \sqrt{(\cos\beta)^2 + (\sin\beta)^2}$, $\|\vec{U}\| = \sqrt{(\cos\alpha)^2 + (\sin\alpha)^2}$

$\|\vec{u}\|\|\vec{U}\| = \sqrt{(\cos\beta)^2 + (\sin\beta)^2}\sqrt{(\cos\alpha)^2 + (\sin\alpha)^2} = (1)(1) = 1$

$\vec{u} \cdot \vec{U} = \cos\theta$

$$\frac{\vec{u} \cdot \vec{U}}{\|\vec{u}\|\|\vec{U}\|} = \frac{\cos\theta}{1} = \cos\theta$$

Sec 1.3:

1. Consider the following vectors and matrices

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \qquad \vec{u} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \end{bmatrix}$$

Compute the following vector-matrix products.

(a) $A\vec{u}$

$$A\vec{u} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \end{bmatrix}$$

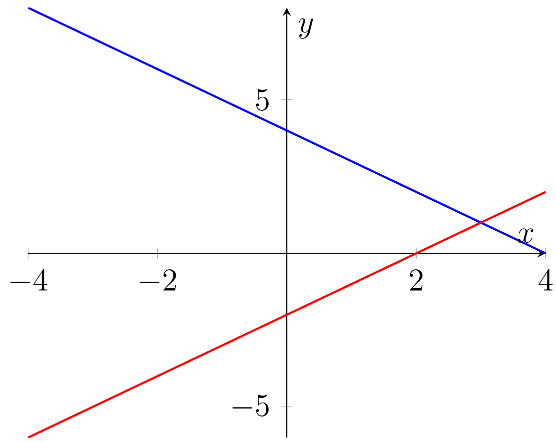
(b) $B\vec{v}$

$$B\vec{v} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} (1,2,3) \cdot (1,2,-1) \\ (3,1,0) \cdot (1,2,-1) \end{bmatrix} = \begin{bmatrix} 1+4-3 \\ 3+2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

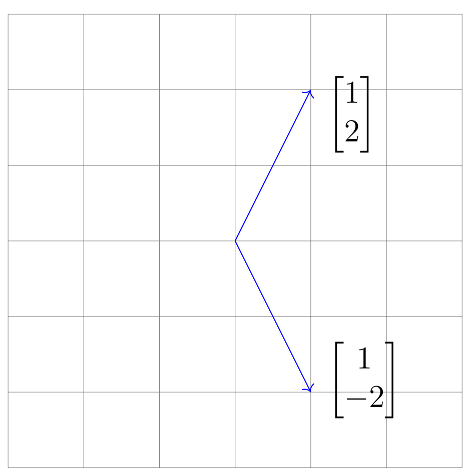
2. Consider the system of equations:

$$x + y = 4 \quad \text{(1a)}$$
$$2x - 2y = 4. \quad \text{(1b)}$$

(a) Draw the row picture (two intersecting lines).



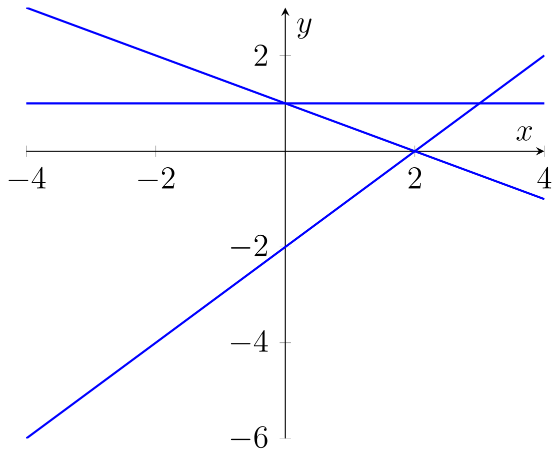
(b) Draw the column picture (combination of two columns equal to the column vector (4,4) on the right side).



$$\begin{bmatrix} x+y \\ 2x-2y \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}, x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

3. Sketch these three lines and decide if the equations are solvable:

$$x + 2y = 2 \tag{2a}$$
$$x - y = 2. \tag{2b}$$
$$y = 1. \tag{2c}$$



$x - 1 = 2 \Rightarrow x = 3 \Rightarrow x + 2y = 3 + 2 = 5$, but in the given equations above $x + 2y = 2$. This is a contradiction, so no solution exists. Graphically, we can see that there is no point of intersection between all three lines.

4. Find A^{-1} by rewriting the following matrix-vector system

$$A\vec{x} = \vec{b} \qquad \Longrightarrow \qquad \begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} x_2 + 2x_3 \\ x_1 - x_2 \\ x_3 \end{bmatrix}$$

Then:

$x_3 = b_3$
$x_2 = b_1 - 2b_3$
$x_1 = b_2 + b_1 - 2b_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 + b_2 - 2b_3 \\ b_1 - 2b_3 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

5. Write the following system of equations as a matrix-vector system.

   Hint: Write $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ so that $A\vec{x} = \vec{b}$.

$$\begin{aligned} 2x + 2y &= 9 \\ -y + z &= 1 \\ x + 6z &= 0 \end{aligned}$$

$$\begin{bmatrix} 2 & 2 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ 0 \end{bmatrix}$$

6. Write the following matrix-vector system as a system of linear equations

$$\begin{bmatrix} 1 & 5 & 2 \\ 3 & 0 & -1 \\ 8 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 2 \\ 3 & 0 & -1 \\ 8 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 5x_2 + 2x_3 \\ x_1 - x_3 \\ 8x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 9 \end{bmatrix}$$

$x + 5y + 2z = -2$

$x - z = 7$

$8x + 2y = 9$