Sec 8.1, 8.2:

1. Let $p(x) \in \mathbb{P}_3$, and consider the subset S of $\mathbb{P}_3$ with $\int_0^1 p(x)dx = 0$. Verify that S is a subspace of $\mathbb{P}_3$, and find its basis, β_S.

Solution: (1) Note that the polynomial $p_0(x) \equiv 0$ satisfies

$$\int_0^1 p_0(x)dx = 0.$$

Therefore $p_0(x) \in S$.

(2) If $p(x), q(x) \in S$, then

$$\int_0^1 (p(x) + q(x))dx = \int_0^1 p(x)dx + \int_0^1 q(x)dx = 0.$$

Therefore $p(x) + q(x) \in S$.

(3) If $c \in \mathbb{R}$ and $p(x) \in S$, then

$$\int_0^1 cp(x)dx = c\int_0^1 p(x)dx = 0.$$

Therefore $cp(x) \in S$. (1), (2) and (3) conclude that S is a subspace of $\mathbb{P}_3$.

To show a basis of S, note that

$$\int_0^1 xdx = \frac{1}{2}, \int_0^1 x^2dx = \frac{1}{3}, \int_0^1 x^3dx = \frac{1}{4}.$$

Then

$$\int_0^1 \left(x - \frac{1}{2}\right) dx = 0, \int_0^1 \left(x^2 - \frac{1}{3}\right) dx = 0, \int_0^1 \left(x^3 - \frac{1}{4}\right) dx = 0.$$

Since $x - \frac{1}{2}$, $x^2 - \frac{1}{3}$ and $x^3 - \frac{1}{4}$ are linearly independent in $\mathbb{P}_3$, then these polynomials are linearly independent in S. To show that they form a base, let $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \in S$. Since

$$0 = \int_0^1 p(x)dx = \int_0^1 a_0 + a_1x + a_2x^2 + a_3x^3dx = a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4}.$$

This implies

$$a_0 = -\frac{a_1}{2} - \frac{a_2}{3} - \frac{a_3}{4}.$$

Therefore

$$p(x) = a_1\left(x - \frac{1}{2}\right) + a_2\left(x^2 - \frac{1}{3}\right) + a_3\left(x^3 - \frac{1}{4}\right),$$

Thus, the polynomials $x - \frac{1}{2}$, $x^2 - \frac{1}{3}$ and $x^3 - \frac{1}{4}$ generate S. The basis is

$$\left\{x - \frac{1}{2}, x^2 - \frac{1}{3}, x^3 - \frac{1}{4}\right\}.$$

2. Prove that T^2 is a linear transformation if T is a linear transformation.

   Solution: Let V be the vector space on $\mathbb{R}$. For a fixed $\boldsymbol{u}, \boldsymbol{v} \in V$ and $c \in \mathbb{R}$,

   $$T^2(\boldsymbol{u}+\boldsymbol{v}) = T(T(\boldsymbol{u}+\boldsymbol{v})) = T(T(\boldsymbol{u})+T(\boldsymbol{v})) = T(T(\boldsymbol{u}))+T(T(\boldsymbol{v})) = T^2(\boldsymbol{u})+T^2(\boldsymbol{v}).$$

   and

   $$T^2(c\boldsymbol{u}) = T(T(c\boldsymbol{u})) = T(cT(\boldsymbol{u})) = cT(T(\boldsymbol{u})) = cT^2(\boldsymbol{u}).$$

   Therefore, T^2 is a linear transformation.

3. Which of the following transformations is not linear? Justify your answer. The input vector is $\boldsymbol{v} = (v_1, v_2)$.

   (a) $T(\boldsymbol{v}) = (v_2, v_1)$.

   Solution: T is linear. For $\boldsymbol{u} = (u_1, u_2)$, $\boldsymbol{v} = (v_1, v_2)$ and $c \in \mathbb{R}$,

   $$T(\boldsymbol{u}+\boldsymbol{v}) = T(u_1+v_1, u_2+v_2) = (u_2+v_2, u_1+v_1) = (u_2, u_1)+(v_2, v_1) = T(\boldsymbol{u})+T(\boldsymbol{v}),$$
   $$T(c\boldsymbol{u}) = (cu_2, cu_1) = c(u_2, u_1) = cT(\boldsymbol{u}).$$

   (b) $T(\boldsymbol{v}) = (0, v_1)$.

   Solution: T is linear. For $\boldsymbol{u} = (u_1, u_2)$, $\boldsymbol{v} = (v_1, v_2)$ and $c \in \mathbb{R}$,

   $$T(\boldsymbol{u}+\boldsymbol{v}) = (0, u_1+v_1) = (0, u_1)+(0, v_1) = T(\boldsymbol{u})+T(\boldsymbol{v}),$$
   $$T(c\boldsymbol{u}) = (0, cu) = c(0, u) = cT(\boldsymbol{u}).$$

   (c) $T(\boldsymbol{v}) = (v_1, v_1)$.

   Solution: T is linear. For $\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{v} = (v_1, v_2)$ and $c \in \mathbb{R}$,

   $$T(\boldsymbol{u}+\boldsymbol{v}) = T[(u_1+v_1, u_2+v_2)] = (u_1+v_1, u_1+v_1) = (u_1, u_1)+(v_1, v_1) = T(\boldsymbol{u})+T(\boldsymbol{v})$$
   $$T(c\boldsymbol{u}) = (cu_1, cu_1) = c(u_1, u_1) = cT(\boldsymbol{u}).$$

   (d) $T(\boldsymbol{v}) = (0, 1)$.

   Solution: T is NOT linear. Choose $c = 2$ and $\boldsymbol{u} = (1, 1)$.

   $$T(2\boldsymbol{u}) = (0, 1) \neq (0, 2) = 2T(\boldsymbol{u}).$$

4. Prove whether the following transformations are linear or not.

   (a) $T(\boldsymbol{v}) = \dfrac{\boldsymbol{v}}{||\boldsymbol{v}||}$.

   Solution: T is NOT linear. Choose $c = 2$ and any $v \neq 0$.

$$T(2\boldsymbol{v}) = \frac{2\boldsymbol{v}}{||2\boldsymbol{v}||} = \frac{2\boldsymbol{v}}{2||\boldsymbol{v}||} = \frac{\boldsymbol{v}}{||\boldsymbol{v}||} = T(\boldsymbol{v}) \neq 2T(\boldsymbol{v}).$$

   (b) $T(v_1, v_2, v_3) = (v_1, 2v_2, 3v_3)$.

   Solution: T is linear. For $\boldsymbol{u} = (u_1, u_2)$, $\boldsymbol{v} = (v_1, v_2)$ and $c \in \mathbb{R}$,

$$\begin{aligned} T(\boldsymbol{u} + \boldsymbol{v}) &= (u_1 + v_1, 2(u_2 + v_2), 3(u_3 + v_3)) \\ &= (u_1 + v_1, 2u_2 + 2v_2, 3u_3 + 3v_3) \\ &= (u_1, 2u_2, 3u_3) + (v_1, 2v_2, 3v_3) \\ &= T(\boldsymbol{u}) + T(\boldsymbol{v}) \end{aligned}$$

   and

$$T(c\boldsymbol{u}) = (cv_1, 2cv_2, 3cv_3) = c(v_1, 2v_2, 3v_3) = cT(\boldsymbol{u}).$$

   (c) $T(v_1, v_2, v_3) = v_1 + v_2 + v_3$.

   Solution: T is linear. For $\boldsymbol{u} = (u_1, u_2)$, $\boldsymbol{v} = (v_1, v_2)$ and $c \in \mathbb{R}$,

$$\begin{aligned} T(\boldsymbol{u} + \boldsymbol{v}) &= u_1 + v_1 + u_2 + v_2 + u_3 + v_3 \\ &= (u_1 + u_2 + u_3) + (v_1 + v_2 + v_3) \\ &= T(\boldsymbol{u}) + T(\boldsymbol{v}). \end{aligned}$$

5. Suppose a linear T transforms $(1,1)$ to $(2,2)$ and $(2,0)$ to $(0,0)$. Find $T(\boldsymbol{v})$ when

(a) $\boldsymbol{v} = (2,2)$

Solution: The 2×2 matrix $[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is such that

$$[T]\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \text{and} \quad [T]\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus,

$$\begin{cases} a+b=2 \\ 2a=0 \end{cases} \quad \text{and} \quad \begin{cases} c+d=2 \\ 2c=0 \end{cases}.$$

We have $[T] = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}$.

$$T(2,2) = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}.$$

Therefore $T(2,2) = (4,4)$.

(b) $\boldsymbol{v} = (3,1)$

Solution: We have

$$T(3,1) = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

. Therefore $T(3,1) = (2,2)$

(c) $\boldsymbol{v} = (-1,1)$

Solution: We have

$$T(-1,1) = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Therefore $T(-1,1) = (2,2)$.

6. Consider the following bases for $\mathbb{R}^2$:

$$\beta_V = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}, \quad \beta_W = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$$

(a) Write the vector $\boldsymbol{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ (given in the standard basis) in terms of the input basis β_V. These coordinates are d_1 and d_2.

Solution:

We want to find d_1, d_2 such that

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} d_1 + \begin{bmatrix} 2 \\ 3 \end{bmatrix} d_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Let

$$V = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}.$$

Thus

$$V\boldsymbol{d} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = V^{-1}\boldsymbol{x} = \begin{bmatrix} 3/5 & -2/5 \\ 1/5 & 1/5 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Therefore $d_1 = -1$ and $d_2 = 1$.

(b) Find the change of basis matrix that will find the coordinates of $\boldsymbol{x}$ in the output basis β_W using the formula $V\boldsymbol{d} = W\boldsymbol{c}$. What are the new coordinates of $\boldsymbol{x}$ in the output basis?

Solution:

$$V\boldsymbol{d} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = W\boldsymbol{c}$$

Therefore,

$$\boldsymbol{c} = W^{-1}V\boldsymbol{d},$$

and thus, the change of basis matrix is

$$W^{-1}V = \begin{bmatrix} 3/5 & 1/5 \\ -2/5 & 1/5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2/5 & 9/5 \\ -3/5 & -1/5 \end{bmatrix}$$

The new coordinates of $\boldsymbol{x}$ are

$$\begin{bmatrix} 2/5 & 9/5 \\ -3/5 & -1/5 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7/5 \\ 2/5 \end{bmatrix}.$$

(c) Consider $\boldsymbol{y} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ using coordinates in the basis β_V. Use your matrix from Part (b) to write $\boldsymbol{y}$ in terms of the output basis β_W.

Solution:

$$\boldsymbol{y}_W = \begin{bmatrix} 2/5 & 9/5 \\ -3/5 & -1/5 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -1/5 \\ -11/5 \end{bmatrix}$$

7. **Change of basis.**

(a) What matrix transforms $\begin{bmatrix}1\\0\end{bmatrix}$ to $\begin{bmatrix}2\\5\end{bmatrix}$ and transforms $\begin{bmatrix}0\\1\end{bmatrix}$ to $\begin{bmatrix}1\\3\end{bmatrix}$?

Solution:

$$[T] = \begin{bmatrix}2 & 1\\5 & 3\end{bmatrix}$$

(b) What matrix transforms $\begin{bmatrix}2\\5\end{bmatrix}$ to $\begin{bmatrix}1\\0\end{bmatrix}$ and transforms $\begin{bmatrix}1\\3\end{bmatrix}$ to $\begin{bmatrix}0\\1\end{bmatrix}$?

Solution: It is the inverse ot $[T]$ in item (a).

$$M = \begin{bmatrix}3 & -1\\-5 & 2\end{bmatrix}$$

(c) Why does no matrix transform $\begin{bmatrix}2\\6\end{bmatrix}$ to $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\3\end{bmatrix}$ to $\begin{bmatrix}0\\1\end{bmatrix}$?

Solution: Because $\begin{bmatrix}2\\6\end{bmatrix}$ and $\begin{bmatrix}1\\3\end{bmatrix}$ are not linearly independent.

(d) What matrix transforms $\begin{bmatrix}2\\5\end{bmatrix}$ to $\begin{bmatrix}1\\1\end{bmatrix}$ and transforms $\begin{bmatrix}1\\3\end{bmatrix}$ to $\begin{bmatrix}0\\2\end{bmatrix}$?

Solution: Let

$$M = \begin{bmatrix}a & b\\ c & d\end{bmatrix}.$$

We have $M\begin{bmatrix}2\\5\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$ and $M\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix}$.

$$\begin{cases} 2a + 5b = 1 \\ 2c + 5d = 1 \\ a + 3b = 0 \\ c + 3d = 2 \end{cases}$$

Thus $a = 3$, $b = -1$. $c = -7$, $d = 3$.

$$M = \begin{bmatrix}3 & -1\\ -7 & 3\end{bmatrix}.$$