Sec 6.4, 6.5:

1. Diagonalize the following symmetric matrix:

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}.$$

Solution: We have

$$|\lambda I - A| = \begin{vmatrix} \lambda - 3 & 2 & -4 \\ 2 & \lambda - 6 & -2 \\ -4 & -2 & \lambda - 3 \end{vmatrix} = (\lambda + 2)(\lambda - 7)^2.$$

Thus, the eigenvalues of A are,

$$\lambda_1 = -2, \quad \lambda_2 = \lambda_3 = 7.$$

Solve $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ separately for $\lambda_1 = -2$, $\lambda_2 = 7$ and $\lambda_3 = 7$:

$(A + 2I)\boldsymbol{x} = \begin{bmatrix} -5 & 2 & -4 \\ 2 & -8 & -2 \\ -4 & -2 & -5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ yields an eigenvector $\boldsymbol{x_1} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ for $\lambda_1 = -2$,

$(A - 7I)\boldsymbol{x} = \begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ yields two eigenvectors $\boldsymbol{x_2} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ and $\boldsymbol{x_3} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ for $\lambda_2 = \lambda_3 = 7$.

Note that $\boldsymbol{x_2}$ and $\boldsymbol{x_3}$ are not orthogonal, so we use G-S to make them orthogonal and we get: $\boldsymbol{x_2} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ and $\boldsymbol{x_3} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$.

It's clear that $\boldsymbol{x_1}$, $\boldsymbol{x_2}$ and $\boldsymbol{x_3}$ are orthogonal but not yet orthonormal. Divide these eigenvectors by their length to get unit vectors,

$$\boldsymbol{x_1} = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}, \quad \boldsymbol{x_2} = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \\ 0 \end{bmatrix}, \quad \textit{and} \quad \boldsymbol{x_3} = \begin{bmatrix} 4/\sqrt{45} \\ 2/\sqrt{45} \\ 5/\sqrt{45} \end{bmatrix}.$$

Then put these unit vectors to the columns of Q,

$$Q = \begin{bmatrix} 2/3 & 1/\sqrt{5} & 4/\sqrt{45} \\ 1/3 & -2/\sqrt{5} & 2/\sqrt{45} \\ -2/3 & 0 & 5/\sqrt{45} \end{bmatrix},$$

and diagonalize matrix A by

$$\Lambda = Q^T A Q = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}.$$

2. Decide whether the following matrices are positive definite, positive semi-definite, or neither.

   (a) $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

   Solution: We have

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & 1 & 1 \\ 1 & \lambda - 2 & 1 \\ 1 & 1 & \lambda - 2 \end{vmatrix} = \lambda(\lambda - 3)^2.$$

   Thus, the eigenvalues of A are,

$$\lambda_1 = 0, \quad \lambda_2 = \lambda_3 = 3.$$

   All eigenvalues of A are greater than or equal to 0, which means that matrix A is positive semi-definite.

   (b) $B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

   Solution: We have

$$|\lambda I - B| = \begin{vmatrix} \lambda - 2 & 1 & 1 \\ 1 & \lambda - 2 & -1 \\ 1 & -1 & \lambda - 2 \end{vmatrix} = (\lambda - 4)(\lambda - 1)^2.$$

   Thus, the eigenvalues of A are,

$$\lambda_1 = 4, \quad \lambda_2 = \lambda_3 = 1.$$

   All eigenvalues of B are more than 0, which means that matrix A is positive definite.

   (c) $C = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}^2$

   Solution: Set $D = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ and no zero vector $\boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. We have

$$x^T C x = x^T \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} x = (Dx)^T Dx = (2x+y)^2 + (y+2z)^2 + (x+z)^2 > 0$$

   It means that matrix C is positive definite.

3. Suppose each "Gibonacci" number G_{k+2} is the average of the two previous numbers G_{k+1} and G_k. Then $G_{k+2} = \frac{1}{2}(G_{k+1} + G_k)$:

   (a) Starting with initial conditions $G_0 = 0$ and $G_1 = 1$, list the elements up to G_6 of the sequence.
   Solution: We have

$$G_2 = 1/2, G_3 = 3/4, G_4 = 5/8, G_5 = 11/16, G_6 = 21/32.$$

   (b) Express the given sequence as $\mathbf{u}_{k+1} = A\mathbf{u}_k$. Clearly indicate the matrix A, as well as the vectors $\mathbf{u}_{k+1}$, $\mathbf{u}_k$, and the vector of initial conditions $\mathbf{u}_0$.
   Solution: Set

$$\mathbf{u}_k = \begin{bmatrix} G_k \\ G_{k+1} \end{bmatrix}, \mathbf{u}_{k+1} = \begin{bmatrix} G_{k+1} \\ G_{k+2} \end{bmatrix},$$

   so

$$\mathbf{u}_{k+1} = A\mathbf{u}_k, \mathbf{u}_0 = \begin{bmatrix} G_0 \\ G_1 \end{bmatrix},$$

   Where $A = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$.

   (c) Find a formula for $\mathbf{u}_{k+1}$ in terms of the initial conditions $\mathbf{u}_0$. *Hint*: Diagonalize A.
   Solution: We have

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ -1/2 & \lambda - 1/2 \end{vmatrix} = (\lambda + 1/2)(\lambda - 1).$$

   Thus, the eigenvalues of A are,

$$\lambda_1 = -1/2, \quad \lambda_2 = 1.$$

   Further more, the eigenvectors are $\boldsymbol{x_1} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\boldsymbol{x_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ with corresponding eigenvalues. So put the eigenvectors to the columns of Q,

$$Q = \begin{bmatrix} -2 & -1 \\ 1 & -1 \end{bmatrix},$$

   and substitute $A = Q\Lambda Q^{-1}$ into iteration equation,

$$\mathbf{u}_{k+1} = A\mathbf{u}_k = A^k\mathbf{u}_0 = Q\Lambda^k Q^{-1}\mathbf{u}_0 = \begin{bmatrix} -2 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} (-1/2)^k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1/3 & 1/3 \\ -1/3 & -2/3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Sec 7.1, 7.2:

1. Consider the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}.$$

(a) Compute A^TA, and find its eigenvalues λ_1 and λ_2, such that $\lambda_1 > \lambda_2$. What are the singular values? What is the matrix Σ, such that $A = U\Sigma V^T$.

Solution: Set

$$A^TA = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 20 \\ 20 & 80 \end{bmatrix},$$

and we have the eigenvalues of A^TA are,

$$\lambda_1 = 85, \quad \lambda_2 = 0.$$

The singular values $\sigma_1 = \sqrt{85}$ and $\sigma_2 = 0$, and $\Sigma = \begin{bmatrix} \sqrt{85} & 0 \\ 0 & 0 \end{bmatrix}$.

(b) Compute the unit eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, which correspond to eigenvalues λ_1 and λ_2, respectively. Which matrix of the SVD do these column vectors correspond to?

Solution: Solve $(A^TA - \lambda I)\boldsymbol{v} = \mathbf{0}$ separately for $\lambda_1 = 85$ and $\lambda_2 = 0$:

$(A^TA - 85I)\boldsymbol{v_1} = \begin{bmatrix} -80 & 20 \\ 20 & -5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ yields an eigenvector $\boldsymbol{v_1} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1/\sqrt{17} \\ 4/\sqrt{17} \end{bmatrix}$ for $\lambda_1 = 85$,

$(A^TA - 0I)\boldsymbol{v_2} = \begin{bmatrix} 5 & 20 \\ 20 & 80 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ yields an eigenvector $\boldsymbol{v_2} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 4/\sqrt{17} \\ -1/\sqrt{17} \end{bmatrix}$ for $\lambda_2 = 0$,

It's clear that $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ are linearly independent unit eigenvectors of A^TA.
In addition,

$$A^TA = (U\Sigma V^T)^T(U\Sigma V^T) = V\Sigma U^TU\Sigma V^T = V\Sigma^2V^T$$

Thus, the column vectors about eigenvectors correspond to the matrix V of SVD.

2. Consider the matrix
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

(a) Compute $A^T A$ and find its eigenvalues.
Solution: Set
$$A^T A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$
and we have
$$|\lambda I - A^T A| = \begin{vmatrix} \lambda - 1 & -1 & 0 \\ -1 & \lambda - 2 & -1 \\ 0 & -1 & \lambda - 1 \end{vmatrix} = \lambda(\lambda - 1)(\lambda - 3).$$
Thus, the eigenvalues of $A^T A$ are,
$$\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 3.$$

(b) Compute AA^T and find its eigenvalues.
Solution: Set
$$AA^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$
and we have
$$|\lambda I - AA^T| = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 3).$$
Thus, the eigenvalues of AA^T are,
$$\lambda_1 = 1, \quad \lambda_2 = 3.$$

(c) Compute the eigenvectors of A^TA.
Solution: Solve $(A^TA - \lambda I)\boldsymbol{v} = \mathbf{0}$ separately for $\lambda_1 = 3$, $\lambda_2 = 1$and $\lambda_3 = 0$:

$\lambda_1 = 3$: $(A^TA - 3I)\boldsymbol{v_1} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ yields an eigenvector

$\boldsymbol{v_1} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$.

$\lambda_2 = 1$: $(A^TA - I)\boldsymbol{v_2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ yields an eigenvector

$\boldsymbol{v_2} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$.

$\lambda_3 = 0$: $(A^TA - 0I)\boldsymbol{v_3} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ yields an eigenvector

$\boldsymbol{v_3} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$.

It's clear that $\boldsymbol{v_1}$, $\boldsymbol{v_2}$ and $\boldsymbol{v_3}$ are linearly independent eigenvectors of A^TA.

(d) Compute the eigenvectors of AA^T.
Solution: The eigenvectors of AA^T are the same as the left singular vectors $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$.

$u_1 = \frac{1}{\sigma_1} A\boldsymbol{v_1} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$u_2 = \frac{1}{\sigma_1} A\boldsymbol{v_2} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

It's clear that $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$ are linearly independent eigenvectors of AA^T.

(e) Find orthonormal bases for the four fundamental subspaces of A.

Solution: Combine (c) and (d), the following results can be showed.

For the column space, the orhomnormal bases are $\boldsymbol{u_1} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, $\boldsymbol{u_2} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$.

For the left nullspace, the only element in this space is $\mathbf{0}$.

For the row space, the orhomnormal bases are $\boldsymbol{v_1} = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$, $\boldsymbol{v_2} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$.

For the nullspace, the orhomnormal bases is $\boldsymbol{v_3} = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$.

(f) Write $A = U\Sigma V^T$.

Solution: Combine (b), (c) and (d), and set the following matrices.

$$U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}, V = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \textit{ and } \quad \Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Thus $A = U\Sigma V^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} =$
$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.