Linear Algebra Cheat Sheet
Inverse Matrix

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1 Properties of Inverse Matrix

- $AA^{-1} = I$, $A^{-1}A = I$
- $(A^T)^{-1} = (A^{-1})^T$
- $(AB)^{-1} = B^{-1}A^{-1}$

Question 1 If A and M have inverse matrix A^{-1} and M^{-1} and

- $AX = B$
- $YM = C$
- $AZM^\top = D$

what is X, Y, Z?

2 Elimination

Elimination as Matrix Operation We can write the operations to change equivalent linear system by $[A|b] \to [E_{ij}A|E_{ij}b]$ and $[P_{ij}A|P_{ij}b]$.

- Elimination matrix E_{ij}:
  - Replace row (j) by $*$*row(i) + row (j)
  - Identity matrix except $a_{ij} = *$
- Permutation matrix P_{ij}:
  - Swtich Row (i) with Row (j)
  - Identity matrix except $a_{ij} = a_{ji} = 1, a_{ii} = a_{jj} = 0$

Question 2 What is the matrix after the following operations

- Change Row 2 of A to Row 2 + 2* Row 1
- Switch Row 3 and Row 4 of the new matrix
- Change Row 4 of the new matrix to Row 4 + 2* Row 2

3 Inverse Matrix

- The inverse of a matrix exists if and only if the matrix is a square matrix and all column vectors are linear independent.
- The inverse of a matrix exists if and only if elimination produced n non-zero pivots.

Questions (answer is in the slide) Can you describe how the upper triangular form and their pivots look like for the following three cases

- The linear system have a single solution
- The linear system have no solution
- The linear system have infinite solutions

Questions Please ensure you know the answer of the following questions

- How to calculate the inverse of a matrix?
- What is the inverse of the elimination matrix? What is the inverse of the permutation matrix?

① Single Solution. n non-zero pivot

$$\left[\begin{array}{cccc|c} * & \times & \cdots & \times & \times \\ & * & & \times & \times \\ & & \ddots & & \vdots \\ & & & * & \times \end{array}\right]$$

*: non-zero
x: any number

② No Solution

pivot is zero → $\left[\begin{array}{ccc|c} 0 & \cdots & 0 & \Delta \end{array}\right]$ ← Δ: non-zero value

$0x_1 + \cdots + 0x_n = \Delta$, not possible!!

③ Infinite Solution.

1. have a row $0 \cdots 0 \mid 0$

2. all the row. whose left part is all zero $0 \cdots 0$ the right part is also 0!!!

$0 \cdots 0 \mid -1$
$0 \cdots 0 \mid 0$
↑ no Solution!!

LU Decomposition!

$$A = L \cdot U$$ (L: lower Triangular; U: upper Triangular)

LDU Decomposition

$$A = L\, D\, U$$

L: Lower Triangular but all 1 on the diag
D: diagonal Matrix
U: upper Triangular but all 1 on the diag

if A is symmetric

$$A = L \cdot D \cdot L^T$$

(LDL decomposition)

1. LDU, LDL are Unique!!
2. LU is not unique.

$A = L_1 U_1 = L_2 U_2$

Then $L_1 = L_2 \cdot D$ (!) (D: diag)

1. LDU Decomposition is Unique!!!

Let's assume $A = L_1 U_1 = L_2 \cdot U_2$

$$L_2^{-1} L_1 U_1 U_1^{-1} = L_2^{-1} L_2 U_2 U_1^{-1}$$

$$\Rightarrow L_2^{-1} \cdot L_1 = U_2 \cdot U_1^{-1}$$

(L_2^{-1}: L.T., $L_2^{-1} \cdot L_1$: L.T.; U_2: U.T., U_1^{-1}: U.T., $U_2 \cdot U_1^{-1}$: U.T.)

① Inverse of Lower Triangular is L.T.
② L.T. × L.T. → L.T.

→ $L_2^{-1} \cdot L_1$ or $U_2 \cdot U_1^{-1}$ are both L.T. and U.T. means. They are diag!!!

$$L_2^{-1} \cdot L_1 = D = U_2 \cdot U_1^{-1}$$ (D: diagonal)

$$L_1 = L_2 \cdot D$$ ($L_2 (L_2^{-1} L_1) = L_2 \cdot D$)

$$U_2 = D \cdot U_1$$ ($D \cdot U_1 = (U_2 \cdot U_1^{-1}) U_1$)

if diag of L_2 and L_1 are 1. the my D is identity

hint!! $L_1 = L_2 \cdot D$

$\Rightarrow (L_1)_{11} = (L_2)_{11} \cdot d_{11} \Rightarrow d_{11} = 1 \Rightarrow$ D is identity!!