Rank and Solvability.

- first of all. for Matrix $A \in \mathbb{R}^{m \times n}$

rank must satisfy . rank $\leq m$. rank $\leq n$.

- Case 1 if rank $= n$. full Column rank (1, 0 Solution)

$\Rightarrow \dim(\text{Nul}(A)) = 0$ which means $\text{Nul}(A) = \{\vec{0}\}$

Recall (all) complete Solution is $\vec{x} = \vec{x}_{spec} + \underline{\vec{x}_{nul}}$ Nul (A) is $\{\vec{0}\}$, if exist special solution, you can only have 1 solution

but you may also don't have a special Solution. $\Rightarrow$ 0 Solution or 1 Solution

example $\begin{cases} x = 1 \\ 2x = 1 \end{cases} \Rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- Case 2 if rank $< n$. $\Rightarrow \dim(\text{Nul}(A)) = n - \text{rank} > 0$ (0, ∞ Solution)

$\Rightarrow \text{Nul}(A) = \text{span}\{\vec{x}_1 \cdots \vec{x}_{n-r}\}$ $\{\vec{x}_1 \cdots \vec{x}_{n-r}\}$ is basis of Nul (A)

Recall (all) complete Solution is $\vec{x} = \vec{x}_{spec} + \vec{x}_{nul}$ $a_1 \cdots a_{n-r} \in \mathbb{R}$ $\Rightarrow \infty$ Solution

① if exist a x_{spec} (special Solution), then $\vec{x}_{spec} + a_1 \vec{x}_1 + \cdots + a_{n-r} \vec{x}_{n-r}$ are all Solution

② There don't exist x_{spec}. example $\begin{cases} x + 0y = 1 \\ 2x + 0y = 1 \end{cases} \Rightarrow \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- Case 3 if rank $= m$. Full Row Rank.

$\left.\begin{array}{l} \dim(\text{Col}(A)) = m \\ \text{Col}(A) \subseteq \mathbb{R}^m \end{array}\right\} \Rightarrow \text{Col}(A) = \mathbb{R}^m$

(in $\mathbb{R}^2$, 2-(linear independent) vectors can span whole $\mathbb{R}^2$
Similarly, in $\mathbb{R}^m$, m-(linear independent) vectors can span whole $\mathbb{R}^m$)

$\Rightarrow$ For any $b \in \mathbb{R}^m$. $b \in \text{Col}(A)$ So $Ax = b$ must have a solution (1. ∞ Solution)

Example.

1. rank = m = n.

Fall row rank $\Rightarrow$ at least one solution
Fall Glumn rank $\Rightarrow$ at most one solution
$\Rightarrow$ must have 1 solution

$\Rightarrow$ Matix is invertible.

2. $A \in \mathbb{R}^{4\times 5}$ rank$(A) = 4$

A fall row rank $\Rightarrow$ $Ax=b$ at least one solution,
rank $= 4 < n = 5$ $\Rightarrow$ $Ax=b$ have 0 or ∞ solution
$\Rightarrow$ $Ax=b$ must have ∞ solution.