MATH-UA 140 - Linear Algebra

Midterm, Spring 2024

Name: ________________________ **NetID:** ________________________

While you wait, please read and check ☑ the following boxes:

☐ Unless I have extra time with the Moses Center, the time limit is **75 minutes**.

☐ I am taking this exam because I am a student enrolled in Professor Lu's section during this time. If this is not the case, I will leave the room immediately.

☐ I wrote my name and NetID (e.g. ab1234) at the top of this page.

☐ I will not detach any pages, especially not the scratch pages at the end.

☐ Except for multiple choice questions, I will show my work.

☐ If I need more space for an exercise, I will make a note and continue on one of the scratch pages.

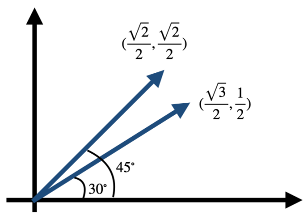
☐ If I am caught in violation of academic integrity, including but not limited to peaking at another student's work, allowing another student to copy from my work, or speaking with another student, or using unauthorized resources, I will be asked to leave the exam and get a zero.

Do not start the exam until you are permitted to.

Exercise I [20 points]

1. Calculate $\cos(75°)$.

   (*hint*: What is the angle between $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$? See the figure below)



   Solution: $\cos(15°) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \cdot \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$ (The two vectors are all unit vectors.)

2. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 4 & 8 & 11 \\ -2 & -4 & -8 & -8 \\ 1 & 1 & 2 & 3 \end{bmatrix}.$$

   Compute the LU factorization of A. Recall that L is lower-triangular with 1s along the diagonal, and U is upper triangular. The matrix U is obtained by bringing A into row echelon form (but not into reduced row echelon form).

   see `https://web.mit.edu/18.06/www/Fall19/Midterm1ReviewSolutions.pdf`

Exercise II [20 points]

For a real number c, consider the linear system

$$\begin{aligned} x_1 + x_2 + cx_3 + x_4 &= c \\ -x_2 + x_3 + 2x_4 &= 0 \\ x_1 + 2x_2 + x_3 - x_4 &= -c \end{aligned}$$

1. For what c, does the linear system have a solution?

   Solution Let us find the REF of the augmented matrix

$$\begin{bmatrix} 1 & 1 & c & 1 & | & c \\ 0 & -1 & 1 & 2 & | & 0 \\ 1 & 2 & -1 & -1 & | & -c \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & c & 1 & | & c \\ 0 & -1 & 1 & 2 & | & 0 \\ 0 & 1 & 1-c & -2 & | & -2c \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & c & 1 & | & c \\ 0 & -1 & 1 & 2 & | & 0 \\ 0 & 0 & 2-c & 0 & | & -2c \end{bmatrix}$$

   Thus the linear system has a solution if and only if $c \neq 2$. 10 points

2. What is the value of c that makes all the solution of the linear system form a vector space?

   solution: $c = 0$ because only $\{x|Ax = 0\}$ can be a vector space. 5 points

3. Find a basis of the subspace of solutions for the value of c from the previous question.

   solution: When $c = 0$, the REF of the unaugmented matrix is

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

   The free variable is x_4 and so solutions are of the form

$$\begin{bmatrix} -3x_4 \\ 2x_4 \\ 0 \\ x_4 \end{bmatrix}$$

   Thus a basis consists of the single vector

$$\begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

   5 points

Exercise III [20 points]

A is a 3×5 matrix. One of your Columbia friends performed row operations on A to convert it to ref form, but did something weird—instead of getting the

$$A \implies \begin{pmatrix} 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 0 & 1 & 0 \\ 6 & 7 & 0 & 0 & 1 \end{pmatrix}.$$

using row operations.

(a) Is it possible to find a basis for Nul(A), Row(A),Col(A).

Since row operations don't change row space and nul space, we can still know Nul(A), Row(A), but we don't know Col(A), computation is similar

(a) Row operations always preserve the null space $N(A)$, i.e. any solution to $Ax = 0$ will be preserved by row operations. Let $H = \begin{pmatrix} F & I \end{pmatrix}$ be the weird row-reduced matrix obtained by our Harvard friend. We can still seek special solutions to $Hx = 0$ using the usual method. Columns 3, 4 and 5 are the pivot columns, while columns 1 and 2 are the free columns. We therefore look for two special solutions:

$$s_1 = \begin{pmatrix} 1 \\ 0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}, s_2 = \begin{pmatrix} 0 \\ 1 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

We can then see that $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ -6 \end{pmatrix}$ and $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -3 \\ -5 \\ -7 \end{pmatrix}$, i.e. the negative entries of each column of F. This gives us a basis for the null space of A:

$$\begin{pmatrix} 1 \\ 0 \\ -2 \\ -4 \\ -6 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \\ -5 \\ -7 \end{pmatrix}.$$

(b) Give a matrix M so that if you multiply A by M (on the left or right?) then the same row operations as the ones used by your Columbia friend will give a matrix in the usual ref form:

(b) We want to first reorder the columns of H so that it is in the usual rref form. Recall that column operations are equivalent to multiplying on the right by an appropriate matrix. A matrix that will put the columns of H in the correct order is the following permutation matrix

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

The matrix $R = HM$ will then be in the usual rref form. Remember that our Harvard friend performed row operations to put our matrix A into the weird form H, and row operations won't change the column order. In particular, recall that row operations are equivalent to multiplying by an appropriate matrix on the left, so there exists a matrix E so that $EA = H$. The product $R = HM = EAM = E(AM)$ is then in the usual rref form. So peforming the same row operations as our Harvard friend on the matrix AMwill give us a matrix in the usual rref form.

Exercise IV [20 points]

1. What is the value of c that makes matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -9 & t \\ 1 & 81 & t^2 \end{bmatrix}$ non-invertible?

   t=1or-9

For parts 2 and 3, circle the right answer. No justification needed.

2. Suppose the matrices A and B have the same column space, then A and B have the same nullspaces.

   A. True B. False

   False

3. There exist a matrix A whose column space is spanned by $(1, 1, 0)$ and $(1, 0, 1)$and whose nullspace is spanned by $(1, 2, 3)$

   A. True B. False

   True

4.If C is any 4 by 7 matrix of rank $r = 3$, find the column space of C. Explain clearly

- can $Cx = b$ have infinitely many solutions. Yes, just provide an example
- can $Cx = b$ have just one solution. No because null space dimension is 4
- can $Cx = b$ have no solution. Yes ,just provide an example

Exercise V [10 points]

Show that all 3×3 matrix whose diagonal sum matries whose diagonal entries average to zero forms a vector space. What is the dimension of the subspace? Provide a basis of the subspace.

Answer: dimension is 8, basis is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Exercise VI [10 points]

$Ax = b_1$ has no solution, and the complete solution to $Ax = b_2$ is:

$$x = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \alpha_1 \begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \end{pmatrix} + \alpha_2 \begin{pmatrix} -2 \\ 0 \\ 1 \\ -2 \end{pmatrix}$$

for all possible scalars α_1 and α_2.

Give **one** example of a possible matrix A (with all nonzero entries) and possible right-hand sides b_1 and b_2 (with any e
A for both b_1 and b_2.)

Hint: start by thinking about possible sizes and ranks for A, and then think how the columns are/aren't linearly dependent.

Answer:

A must have 4 columns, since x is 4×1. Since x is the complete solution to $Ax = b_2$, we know that $N(A)$ has dimension 2. The rank of A is the number of columns minus the dimension of $N(A)$, so the rank of A is 2. This means that A must have at least 2 rows. If A has exactly 2 rows, it would have full row rank and solutions to $Ax = b$ would exist for all b. We don't want this to happen, so we should make A have more than 2 rows. So we will look for a rank 2 A with 4 columns and 3 rows.

Let the columns of A be c_1, c_2, c_3 and c_4. A basis for $N(A)$ is $\{[1\ 1\ 0\ -1], [-2\ 0\ 1\ -2]\}$. We can simply read off a linear relation involving columns 1,2,4 from the vector $[1\ 1\ 0\ -1]$ in $N(A)$. Since

$$A \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} = c_1 + c_2 - c_4 = 0$$

we have that $c_2 = -c_1 + c_4$. We can also read off a linear relation involving columns 1, 3, 4: since

$$A \begin{pmatrix} -2 \\ 0 \\ 1 \\ -2 \end{pmatrix} = -2c_1 + c_3 - 2c_4 = 0$$

we have that $c_3 = 2c_1 + 2c_4$. So, for **any choice** of c_1 and c_4, we can **determine** c_2 and c_3. (In fact these nullspace vectors are exactly in the form of the "special solutions" from class: we can interpret c_2 and c_3 as the "free" columns of A, and c_1 and c_4 as the pivot columns.)

So to choose A, we just need to pick linearly independent vectors c_1, c_4 in $\mathbb{R}^3$ (to make A rank 2!) with nonzero entries as required by the problem (and we need to make sure that c_2 and c_3 also have nonzero entries). One choice is

$$c_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad c_4 = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}.$$

Using the relations we determined above to solve for c_2 and c_3, we get

$$A = \begin{pmatrix} 1 & 2 & 8 & 3 \\ 1 & 1 & 6 & 2 \\ 1 & 1 & 6 & 2 \end{pmatrix}.$$

Now, $b_2 = Ax$, where x is given in the problem. So

$$b_2 = \begin{pmatrix} 1 & 2 & 8 & 3 \\ 1 & 1 & 6 & 2 \\ 1 & 1 & 6 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 41 \\ 29 \\ 29 \end{pmatrix}.$$

Now we find b_1, which is any vector not in the column span of A. Notice that the final two rows of A are identical. This means that the second and third entry of Ax are always equal. That is, a vector

$$y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

cannot be in the column span of A unless $y_2 = y_3$. So any vector with $y_2 \neq y_3$ will work as b_1; for example,

$$b_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

would work.

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