MATH-UA 140 - Linear Algebra

Midterm, Spring 2024

Name: ____________________ **NetID:** ____________________

While you wait, please read and check ☑ the following boxes:

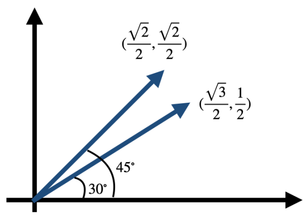
- ☐ Unless I have extra time with the Moses Center, the time limit is **75 minutes**.
- ☐ I am taking this exam because I am a student enrolled in Professor Lu's section during this time. If this is not the case, I will leave the room immediately.
- ☐ I wrote my name and NetID (e.g. ab1234) at the top of this page.
- ☐ I will not detach any pages, especially not the scratch pages at the end.
- ☐ Except for multiple choice questions, I will show my work.
- ☐ If I need more space for an exercise, I will make a note and continue on one of the scratch pages.
- ☐ If I am caught in violation of academic integrity, including but not limited to peaking at another student's work, allowing another student to copy from my work, or speaking with another student, or using unauthorized resources, I will be asked to leave the exam and get a zero.

Do not start the exam until you are permitted to.

Exercise I [20 points]

1. Calculate $\cos(15°)$.

   (*hint*: What is the angle between $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$? See the figure below)



   Solution: $\cos(15°) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \cdot \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \frac{\sqrt{6}+\sqrt{2}}{4}$ (The two vectors are all unit vectors.)

   10 points

2. Calculate the inverse matrix of $\begin{bmatrix} 1 & 0 & -1 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$

   Solution: Given matrix A and the identity matrix I, we start with:

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{array}\right] \quad (1)$$
$$\text{Thus, the inverse matrix } A^{-1} \text{ is:} A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

   10 points, one time of computational error deduct 2 points.

Exercise II [20 points]

1. Use elimination to to put the matrix $A = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ -1 & 2 & -2 & -1 & 2 \\ 0 & -2 & 0 & 0 & -1 \\ 0 & 0 & 0 & -2 & 2 \end{bmatrix}$ in row echelon form.

   Solution: Using Gaussian elimination we get:

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & -1 & 2 \\ 0 & -2 & 0 & 0 & -1 \\ 0 & 0 & 0 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & -1 & 2 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & -1 & 2 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   5 points

2. Write the matrix A as $A = LU$, here L is a lower triangular 4×4 matrix and U is a 4×5 matrix in row echelon form.
   Solution: We can rewrite these steps as multiplications by various elimination matrices and diagonal matrices. The first step is given by $E_{21}^{(1)}$, the second by $E_{32}^{(1)}$, and the third by $E_{43}^{(-2)}$. Thus we get:

$$U = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & -1 & 2 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad L = \left(E_{21}^{(-1)}\right)\left(E_{32}^{(-1)}\right)\left(E_{43}^{(2)}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

   5 points: 2points for writing down E_{ij}, 2points for writing down $E_{ij}^{-}1$, 1 point for L.
   If the students's L is right, he can get all the points.

3. Provide a basis of Col(A).

   Solution: $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ -2 \end{bmatrix} \right\}$ 5 points

4. Provide a basis of Row(A).

   Solution: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$ 5 points

5. Explain why for any 5×4 matrix X, the product AX cannot be invertible.

   (*hint*: What is the rank of matrix A? Use the relation between rank(A) and rank(AX))

   Solution: By part (2), the columns of A are linearly dependent (rank(A) = 3), so they span a vector space of dimension at most $3 < 4$. Since the columns of AX are linear combinations of the columns of A, we conclude that the columns of AX also span vector space of dimension at most $3 < 4$. So AX cannot be invertible, since invertible matrices have full dimensional column space.

Exercise III [20 points]

For a real number c, consider the linear system

$$\begin{aligned} x_1 + x_2 + cx_3 + x_4 &= c \\ -x_2 + x_3 + 2x_4 &= 0 \\ x_1 + 2x_2 + x_3 - x_4 &= -c \end{aligned}$$

1. For what c, does the linear system have a solution?

   Solution Let us find the REF of the augmented matrix

$$\begin{bmatrix} 1 & 1 & c & 1 & | & c \\ 0 & -1 & 1 & 2 & | & 0 \\ 1 & 2 & -1 & -1 & | & -c \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & c & 1 & | & c \\ 0 & -1 & 1 & 2 & | & 0 \\ 0 & 1 & 1-c & -2 & | & -2c \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & c & 1 & | & c \\ 0 & -1 & 1 & 2 & | & 0 \\ 0 & 0 & 2-c & 0 & | & -2c \end{bmatrix}$$

   Thus the linear system has a solution if and only if $c \neq 2$. 10 points

2. What is the value of c that makes all the solution of the linear system form a vector space?

   solution: $c = 0$ because only $\{x | Ax = 0\}$ can be a vector space. 5 points

3. Find a basis of the subspace of solutions for the value of c from the previous question.

   solution: When $c = 0$, the REF of the unaugmented matrix is

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

   The free variable is x_4 and so solutions are of the form

$$\begin{bmatrix} -3x_4 \\ 2x_4 \\ 0 \\ x_4 \end{bmatrix}$$

   Thus a basis consists of the single vector

$$\begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

   5 points

Exercise IV [20 points]

1. The complete solution of linear system $Ax = b$ is $\vec{x} = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 1 \\ 4 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$, then what is dim(Col($A$)),dim(Nul($A$)) and rank($A$)? (You may answer need more information to decide)

   (*hint*: How many columns do A have? How many free variables do this system have?)

   Solution: A have 5 columns and 2 free variables.

   Thus $\dim(\mathrm{Nul}(A)) = 2$,$\mathrm{rank(A)} = n - \dim(\mathrm{Nul}(A)) = 3$ and $\dim(\mathrm{Col}(A)) = \mathrm{rank(A)} = 3$.

   A have 5 columns 1 points, 2 free variable 1 points, every dimension 1 points

   If the student do all the dimension right, he can get full credit.

For parts 2 and 3, circle the right answer. No justification needed.

2. Any elimination matrix size n × n is invertible.

   A. True B. False

   Solution: True 5 points

3. There exist a matrix A whose column space is spanned by $(1, 2, 0)$ and $(2, 2, 1)$and whose nullspace is spanned by $(1, 2, 0, 1)$

   (*hint*: What is the size and rank of the matrix?)

   A. True B. False

   Solution: False 5 points

4.If C is any 4 by 7 matrix of rank $r = 4$, find the column space of C. Explain clearly why $Cx = b$ always has infinitely many solutions.

Solution: The rank of the matrix is equal to the dimension of the column space. Thus the dimension of the column space is 4. Therefore, the column space spans all of R4. A basis of the column space is $[1, 0, 0, 0]$, $[0, 1, 0, 0]$, $[0, 0, 1, 0]$ and $[0, 0, 0, 1]$. Hence, for any vector b there exists a solution. In addition, the dimension of the nullspace is 3. Therefore, there are infinitely many solutions.

full row rank 2.5pt, Nul(A) is 3 dimensional (or dim > 0 is enough) 2.5pt

Exercise V [10 points]

Does all the symmetric matrix $A \in \mathbb{R}^{2\times 2}$ forms a vector space,why? If so, what is the dimension of the space and provide a basis of the space.

Yes! It's a vector space. This is because for any tow symmetric matrix

$$\begin{bmatrix} a_1 & b_1 \\ b_1 & c_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ b_2 & c_2 \end{bmatrix},$$

their linear combination

$$d_1 \begin{bmatrix} a_1 & b_1 \\ b_1 & c_1 \end{bmatrix} + d_2 \begin{bmatrix} a_2 & b_2 \\ b_2 & c_2 \end{bmatrix} = \begin{bmatrix} d_1 a_1 + d_2 a_2 & d_1 b_1 + d_2 b_2 \\ d_1 b_1 + d_2 b_2 & d_1 c_1 + d_2 c_2 \end{bmatrix}$$

is also a symmetric matrix. It's dimension is 3. The basis is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Subspace proof 6pts. using $0 \in V$ $cx \in V$ $x_1 + x_2 \in V$ is also right.
$\dim(V) = 3$ 1 pt
The rank of student's basis:
rank $=1$ 1 pt
rank $=2$ 2 pt
rank $=3$ 3 pt

Exercise VI [10 points]

Find all values of a such that

- $\operatorname{rank}(A) = 1$
- $\operatorname{rank}(A) = 2$
- $\operatorname{rank}(A) = 3$

where

$$A = \begin{bmatrix} 1 & 2 & a \\ -2 & 4a & 2 \\ a & -2 & 1 \end{bmatrix}$$

Solution:Using the elimination method, we obtain:

$$A = \begin{bmatrix} 1 & 2 & a \\ -2 & 4a & 2 \\ a & -2 & 1 \end{bmatrix} \xrightarrow[R_3 - aR_1]{R_2 + 2R_1} \begin{bmatrix} 1 & 2 & a \\ 0 & 4a+4 & 2+2a \\ 0 & -2-2a & 1-a^2 \end{bmatrix} = B$$

Let us consider two cases. (5pt)

Case 1: $a = -1$. Then the matrix B is equal to

$$\begin{bmatrix} 1 & 2 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, B (and hence A) has rank 1. (5pt)

Case 2: $a \neq -1$. Then we divide the second and the third rows of B by $4a+4$ and $-2-2a$ respectively:

$$\begin{bmatrix} 1 & 2 & a \\ 0 & 4a+4 & 2+2a \\ 0 & -2-2a & 1-a^2 \end{bmatrix} \xrightarrow[R_3/(-2-a)]{R_2/(4a+4)} \begin{bmatrix} 1 & 2 & a \\ 0 & 1 & \frac{1}{2} \\ 0 & 1 & \frac{a^2-1}{-2-2a} \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 2 & a \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{a-2}{2} \end{bmatrix} = C.$$

Let us again consider two cases.

Case 2a: $a = 2$. Then $\begin{bmatrix} 1 & 2 & a \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{a-2}{2} \end{bmatrix} = \begin{bmatrix} 1 & 2 & a \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$ has rank 2. (5pt)

Case 2b: $a \neq 2$. Then

$$\begin{bmatrix} 1 & 2 & a \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{a-2}{2} \end{bmatrix} \xrightarrow{R_3/\left(\frac{a-2}{2}\right)} \begin{bmatrix} 1 & 2 & a \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}.$$

The last matrix has rank 3. (5pt)

If student did calculation error in elimination, but all logic below is right. deduct 4 points.

If student have logical error in doing the rank. deduct 4 points.

If student have calculation error in calculating a. deduct 2 points.

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