Linear Transform and Change of Basis

Recap 1. Vector Space V closed respect to linear combination

$V_1, V_2 \in V \rightarrow c_1 V_1 + c_2 V_2 \in V$

Example. 1) $\mathbb{R}^n$ Vector Space is "generalization/abstract definition" of $\mathbb{R}^n$

2) $\{x \mid Ax = 0\}$. $\{Ax \mid x \in \mathbb{R}^n\}$

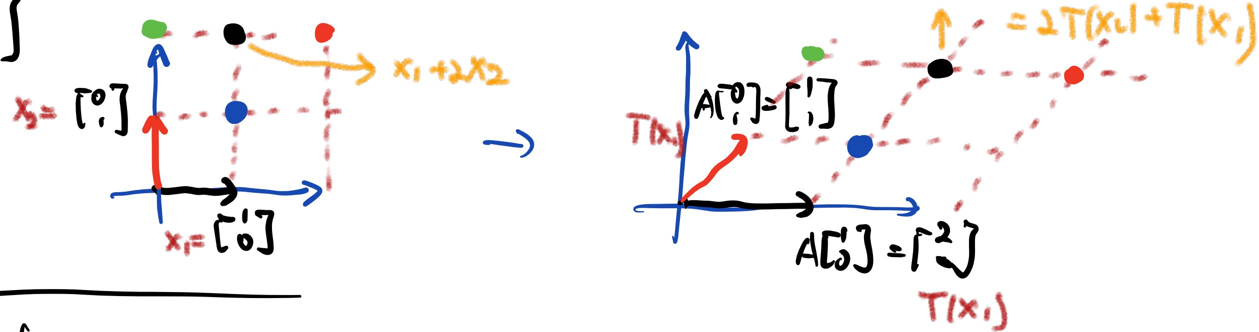
3) $\mathbb{P}_n$: $\mathbb{P}_2 = \{f(x) \mid f(x) = ax^2 + bx + c.\ a.b.c \in \mathbb{R}\}$

$\dim(\mathbb{P}_2) = 3$ basis : $x^2 . x . 1$

"Linear Transform" generalization/abstract definition of Matrixs

Recap 2. consider matrix A as a transform $x \rightarrow Ax$

ex. $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$



Linear Transformation.

$$T(c_1 x_1 + c_2 x_2) = c_1 T(x_1) + c_2 T(x_2) \quad (\Delta) \Rightarrow \text{linear transform}$$

- prove it's a linear transform. check (Δ)
- prove it's not a linear transform give a counter example!

Check (Δ) : ① $T(c \cdot x) = c \cdot T(x)$

② $T(x_1 + x_2) = T(x_1) + T(x_2)$ in Final

Examples. $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ x_1 \\ x_2 + x_1 \end{bmatrix} \Rightarrow T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

check ①. ②

$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 x_2 \\ x_2 \end{bmatrix}$ counter example.

$\vec{u_1} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $\vec{u_2} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ $T(\vec{u_1} + \vec{u_2}) \neq T(\vec{u_1}) + T(\vec{u_2})$

Example. $- T: F \to F$ $\quad Tf = f'$ is a linear tranform

function function.

① $(cf)' = c \cdot f'$ $\quad$ (example $f(x) = x$, $\; c \cdot f(x) = c \cdot x$
$f'(x) = 1 \quad (cf)'(x) = c$)

② $(f_1 + f_2)' = f_1' + f_2'$

Remark. $TT(\sin x) = T((\sin x)') = T(\cos x) = (\cos x)' = -\sin x$

$\sin x$ the eigenvector of $\underline{TT}$

$T = T^T \qquad TT = TT'$

Example. $- T: \mathbb{P}_2 \longrightarrow \mathbb{P}_1$

$\{f(x) \mid f(x) = ax^2 + bx + c\} \qquad \{f(x) \mid f(x) = ax + b\}$

$Tf = f'$ is a linear transform

$f(x) = ax^2 + bx + c \longrightarrow f'(x) = 2ax + b$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \xrightarrow{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}} \begin{bmatrix} 2a \\ b \end{bmatrix}$$

Can you find out matrix

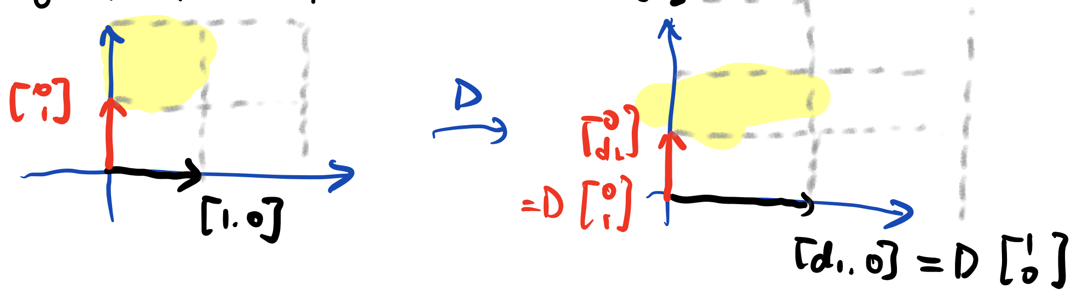
$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\; a = 1, b = 0, c = 0 \quad x^2 \xrightarrow{Tf = f'} 2x = 2 \cdot x + 0 \quad \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\; a = 0, b = 1, c = 0 \quad x \longrightarrow 1 = 0 \cdot x + 1 \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\; a = 0, b = 0, c = 1 \quad 1 \longrightarrow 0 = 0 \cdot x + 0 \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

① Special Linear Transform

1. Diagonal Matrix. $D = \begin{bmatrix} d_1 & \\ & d_2 \end{bmatrix}$



2. Orthogonal Matrix Q

$x \to Qx$

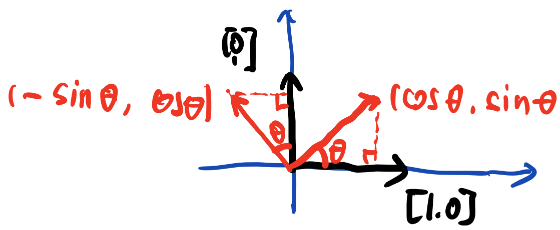
lemma. $x^T y = (Qx)^T (Qy)$ angle will not change

$$= \underbrace{x^T Q^T Q}_{I} y$$

$x^T x = (Qx)^T (Qx)$ the length will not change

$$= x^T \underbrace{Q^T Q}_{I} x$$

Orthogonal : Rotation



$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$

$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$

what is the Matrix of Transform T.

$$T(x) = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\text{an orthogonal Matrix}} x$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

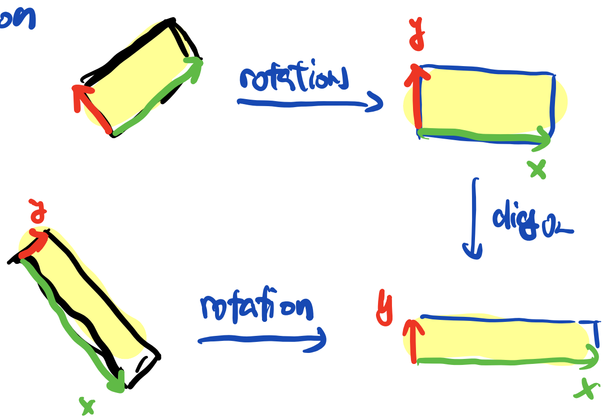
SVD: $A = U \Sigma V^T$

orthogonal means rotation

$$\begin{tikzcd} x \arrow[r, "V^T"] \arrow[d] & V^T x \arrow[d, "\Sigma"] \\ Ax & \Sigma V^T x \arrow[l, "U"] \end{tikzcd}$$

Σ ← diagonal.

Ax
$= U \Sigma V^T x$



How to find a matrix A such that $T(x) = A \cdot x$

- $T\left(\underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{\vec{e_1}}\right) = \begin{bmatrix}1\\2\end{bmatrix}$ $T\left(\underbrace{\begin{bmatrix}0\\1\\0\end{bmatrix}}_{\vec{e_2}}\right) = \begin{bmatrix}3\\4\end{bmatrix}$ $T\left(\underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{\vec{e_3}}\right) = \begin{bmatrix}5\\6\end{bmatrix}$

We call $\vec{e_1}$. $\vec{e_2}$. $\vec{e_3}$ are standard basis of $\mathbb{R}^3$

$T: \mathbb{R}^3 \to \mathbb{R}^2$. $A \in \mathbb{R}^{2\times 3}$

$A = \begin{bmatrix}1 & 3 & 5\\2 & 4 & 6\end{bmatrix} \Rightarrow$ check. $T(\vec{e_1})$ $T(\vec{e_2})$ $T(\vec{e_3})$

If we want to find out matrix A such that

$$T(x) = A \cdot x$$

$T: \mathbb{R}^n \to \mathbb{R}^m$. $A \in \mathbb{R}^{m\times n}$

$$A = [\ T(e_1),\ T(e_2),\ \cdots\cdots,\ T(e^n)\ ]$$

$T(e_1) \in \mathbb{R}^m$ $T(e_2) \in \mathbb{R}^m$ $T(e^n) \in \mathbb{R}^m$.

$T: \mathbb{R}^n \to \mathbb{R}^n$

$\{x_1, \cdots, x_n\}$ is a set of basis

$\{y_1, \cdots, y_n\}$ is another set of basis

Can we find out. $Tx_1 = y_1$

$Tx_2 = y_2$

$\vdots$

$Tx_n = y_n$.

$\underbrace{[y_1 \cdots y_n]}_{n\times n \text{ matrix}} = [T(x_1) \cdots T(x_n)]$

$= [Ax_1 \cdots Ax_n]$

$= A \underbrace{[x_1 \cdots x_n]}_{n\times n \text{ matrix}}$

$\Downarrow$

$A = [y_1 \cdots y_n][x_1 \cdots x_n]^{-1}$

We already answer the question

$\underline{x_1 = e_1}$, $\cdots$ $\underline{x_n = e_n}$

$T: x \to Ax$: $[T(e_1), T(e_2), \cdots T(e_n)]$

<u>Question 2</u>.

$\vec{x} = c_1 \underline{\vec{x_1}} + c_2 \underline{\vec{x_2}} + \cdots + c_n \underline{\vec{x_n}}$

$= d_1 \vec{y_1} + d_2 \vec{y_2} + \cdots + d_n \vec{y}$

Question is. if I know $(c_1 \cdots c_n)$, can we know $(d_1 \cdots d_n)$?.

① $(c_1 \cdots c_n) \to (d_1 \cdots d_n)$ is a linear transform.

$c_1\vec{x_1} + c_2\vec{x_2} + \cdots + c_n\vec{x_n} = d_1\vec{y_1} + d_2\vec{y_2} + \cdots + d_n\vec{y_n}$

$\quad\quad\quad\quad \| \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \|$

$$\underbrace{[\vec{x_1} \quad \vec{x_2} \cdots \vec{x_n}]}_{n\times n \text{ matrix}} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \underbrace{[\vec{y_1} \quad \vec{y_2} \cdots \vec{y_n}]}_{n\times n \text{ matrix}} \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} = [\vec{y_1} \quad \vec{y_2} \cdots \vec{y_n}]^{-1} [\vec{x_1} \quad \vec{x_2} \cdots \vec{x_n}] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Similar Matrix

$$A = X B X^{-1}$$

- Compute the eigenvector of A. $[v_1 \cdots v_n]$
- Compute the eigenvector of B $[u_1 \cdots u_n]$

$$X = [v_1 \cdots v_n][u_1 \cdots u_n]^{-1}.$$

is doing a change of basis. from eigenvector of A to the eigenvector of B

$$\begin{tikzcd}
x \arrow[r, "X^{-1}"', "\text{change of basis}"'] \arrow[d] & X^{-1}x \arrow[d, "B"] \\
Ax & BX^{-1}x \arrow[l, "X"', "\text{Change our basis back.}"]
\end{tikzcd}$$

$$Ax = XBX^{-1}x$$

The same linear transform under different basis!