Linear Algebra

Lecture 5

Inverse Matrices

and LU Decomposition.

Dr. Yiping Lu

Strang Sections 2.5 – Inverse Matrices

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text

The Idea of Inverse Matrices

The idea of Inverse Matrices

• if A have an inverse matrix A^{-1}

This means $Ax = b$ have a single sol'tion $x = A^{-1}b$ $x, b \in \mathbb{R}^n$

square $\mathbb{R}^{n \times n}$

Suppose A is an $n \times n$ matrix (square matrix), then A is invertible if there exists a matrix A^{-1} such that

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I.$$

Matrix Eq $A X = B$

$\mathbb{R}^{n \times n}$ $\mathbb{R}^{n \times p}$

$\mathbb{R}^{n \times p}$

What is X

a) $A^{-1}B$ b) $\cancel{BA^{-1}}$

nxn nxp nxp nxn

We can only talk about an inverse of a square matrix, but not all square matrices are invertible. We will discuss such restrictions in future lectures.

using $AA^{-1} = I$, $A^{-1}A = I$

$$Ax = B$$

$$\underbrace{A^{-1}(Ax)}_{I} = A^{-1}(B)$$

$$\Downarrow$$

$$Ix = A^{-1}B \Rightarrow x = A^{-1}B.$$

$X = A^{-1}B$. $B = [\vec{b}_1 \ \cdots \ \vec{b}_p]$

column vector

$= [A^{-1}\vec{b}_1, \cdots \ A^{-1}\vec{b}_p]$

$Ax_1 = b_1$

$\vdots$

$Ax_p = b_p$

$A X = B$

$A \cdot [A^{-1}\vec{b}_1, \cdots \ A^{-1}\vec{b}_p] = [A\cancel{A^{-1}}b_1, \cdots, A\cancel{A^{-1}}b_p] = B$

The idea of Inverse Matrices

Recall: The multiplicative inverse (or reciprocal) of a nonzero number a is the number b such that $ab = 1$. We define the inverse of a matrix in almost the same way.

Definition

Let A be an $n \times n$ square matrix. We say A is **invertible** (or **nonsingular**) if there is a matrix B of the same size, such that

$$AB = I_n \quad \text{and} \quad BA = I_n.$$

identity matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

In this case, B is the **inverse** of A, and is written A^{-1}.

Example

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

I claim $B = A^{-1}$. Check:

Properties of Inverses

Inverse of a Product

Theorem: If A and B are invertible, then AB is invertible, with

$AB \neq BA$ (!)

$$(AB)^{-1} = B^{-1}A^{-1}$$

!! order

$$\underbrace{(AB)^{-1}}_{\uparrow} \cdot AB = I$$

$$= B^{-1}\underbrace{A^{-1} \cdot A}_{I} \cdot B$$

$$= \underbrace{B^{-1} \cdot B}_{I} = I$$

$$AB \cdot \underline{(AB)^{-1}} = I$$

$$= A\underbrace{B\,B^{-1}}_{I}A^{-1}$$

$$= \underbrace{A\,A^{-1}}_{I} = I$$

Solve the LS

$ABx = y$

$\downarrow$

First Solve Eq $Ax_1 = y$ (1) $\leftarrow$ $x_1 = A^{-1}y$ (Solving (1))

Second step Solve $Bx = x_1$ (2) $\leftarrow$ $x = B^{-1}x_1$

$= B^{-1}(A^{-1}y)$

$\therefore (AB)^{-1} = B^{-1}A^{-1}$

first step Solve $Ax_1 = y$

Second step Solve $Bx = x_1$

Inverse of the sum of Matrices

In general, even if both A and B are invertible matrices of the same size, the matrix $(A+B)$ is not necessarily invertible.

$1 \neq 0$ $\quad (-1) \neq 0$ $\quad 1+(-1) = 0$

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad A+B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Inverse of a Diagonal Matrix

Let $D = \begin{bmatrix} d_{11} & & & \\ & d_{22} & & \\ & & \ddots & \\ & & & d_{nn} \end{bmatrix}$ be an $n \times n$ diagonal matrix, then

$$x \longrightarrow Dx$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad \begin{pmatrix} d_{11}x_1 \\ \vdots \\ d_{nn}x_n \end{pmatrix} \longrightarrow \begin{matrix} /d_{11} \\ \\ /d_{nn} \end{matrix} \rightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$D^{-1} = \begin{bmatrix} 1/d_{11} & & & \\ & 1/d_{22} & & \\ & & \ddots & \\ & & & 1/d_{nn} \end{bmatrix}$ provided that $d_{ii} \neq 0$.

Inverse of an Elimination Matrix

Consider the elimination matrix

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ c & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

a_{31}

which adds c copies of the first row to the third row. Then,

$$E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ -c & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -c & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$x \longrightarrow E_{31} x$

Replace row (3) by

Row (3) + c Row (1)

Row(3')

Row(3)

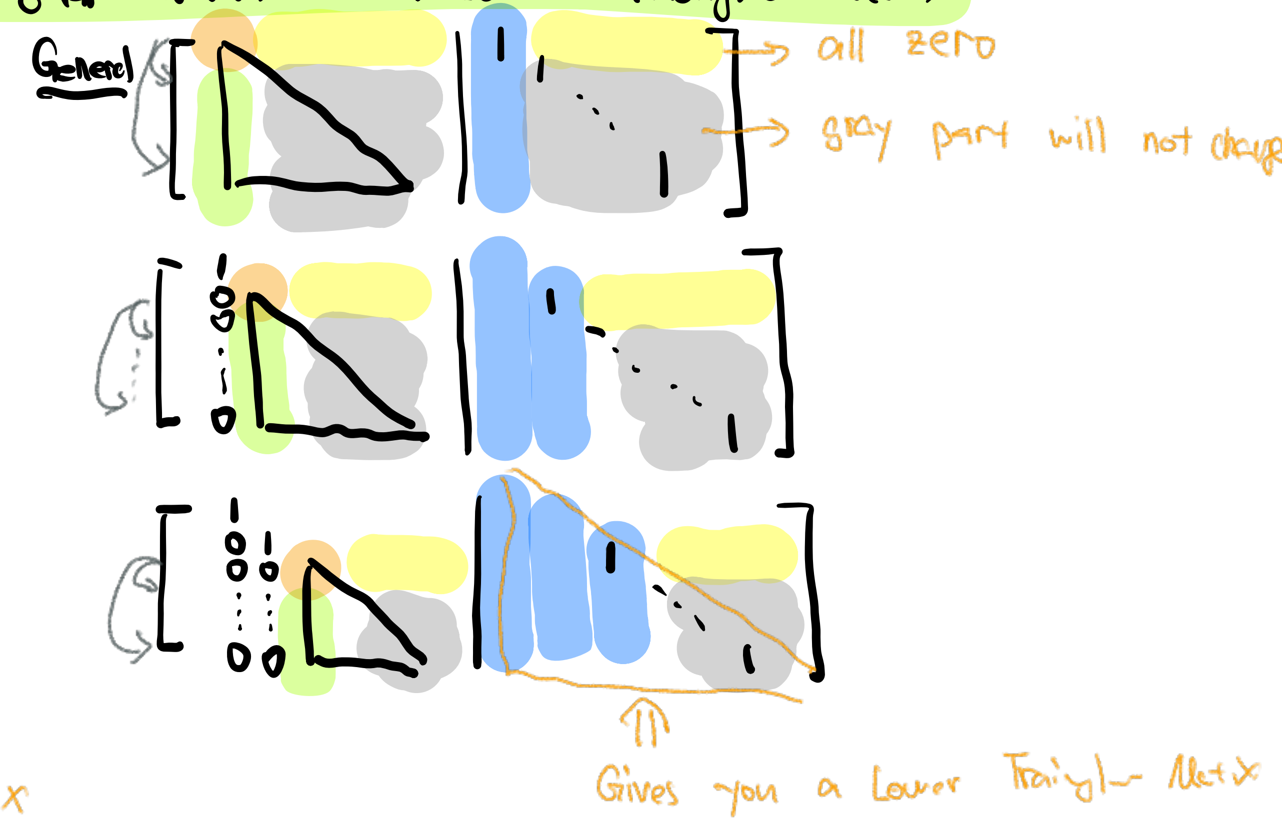
= Row(3') − c Row (1)

another Elimination.

Goal

$E_{31} = \begin{bmatrix} 1 & & \\ & 1 & \\ c & & 1 \end{bmatrix}$ lower traingular $\qquad E_{31}^{-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ -c & & 1 \end{bmatrix}$ lower traingular.

The inverse of a Lower Trainguler Matrix is a Lower Traingular Matrix



Example

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R2 - R1 \\ R3 - R1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] \; R3 - R2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

I_3 Lower Trairgulor Matrix

Inverse of a Permutation Matrix

The inverse of a permutation matrix is its transpose.

just switch two rows

$a_{34} = a_{43} = 1$

$a_{33} = a_{44} = 0$

$x \to P_{ij}x$ switch Row (i) and Row (j)

$$P_{34} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Longrightarrow P_{34}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = P_{34} = P_{34}^T$$

New!!

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \to Px = \begin{pmatrix} x_4 \\ x_2 \\ x_1 \\ x_3 \end{pmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow P^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = P^T$$

x		Px			x	
Row (1)	$\to$	Row (4)	$a_{41} = 1$	$\to$	Row (1)	$a_{14} = 1$
(2)		(2)	$a_{22} = 1$		Row (2)	$a_{22} = 1$
(3)		(1)	$a_{13} = 1$		Row (3)	$a_{31} = 1$
(4)		(3)	$a_{34} = 1$		Row (4)	$a_{43} = 1$

More on the Transpose of a Matrix

Recall

The transpose of an $m \times n$ matrix A is denoted by A^T, and it has entries $a^T_{ij} = a_{ji}$. That is, the columns of A^T are the rows of A.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \implies A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & & & \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

Properties of the Transpose

sum: $(A+B)^T = A^T + B^T$

order!!

product: $(AB)^T = B^T A^T$

$\underbrace{\mathbb{R}^{m\times n} \quad \mathbb{R}^{n\times p}}_{\mathbb{R}^{m\times p}} \rightarrow \mathbb{R}^{p\times m}$

$\underbrace{\mathbb{R}^{p\times n} \quad \mathbb{R}^{n\times m}}_{\mathbb{R}^{p\times m}}$

inverse: $(A^T)^{-1} = (A^{-1})^T$

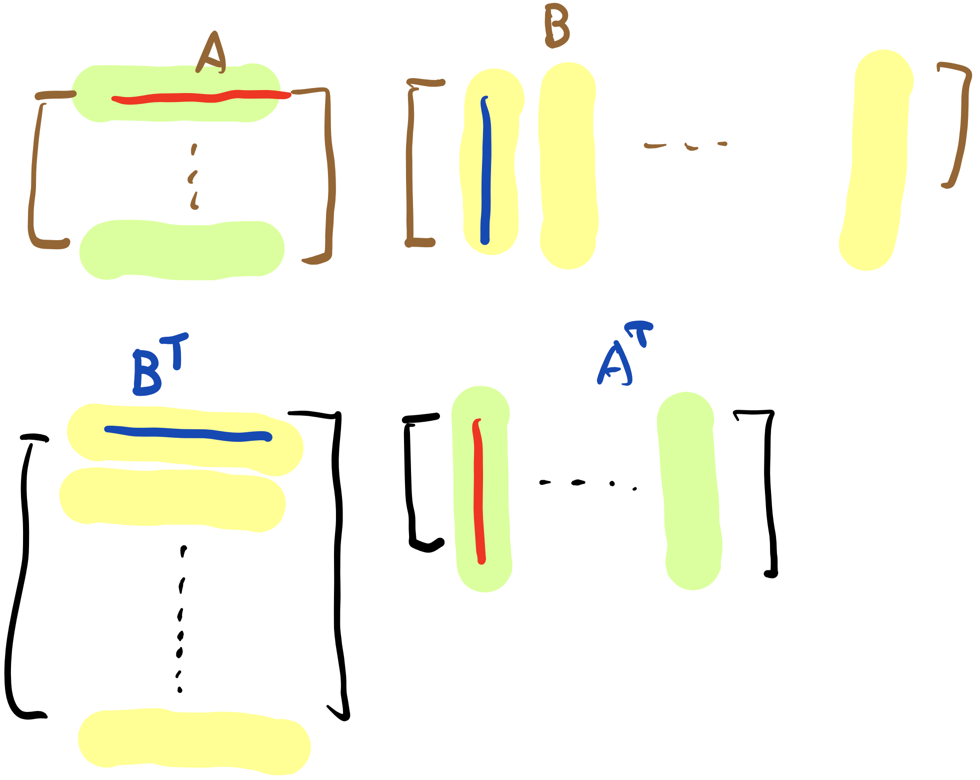
$(A^T)^{-1} \cdot A^T = I.$ $\qquad A^T \cdot (A^T)^{-1} = I$

$= (A^{-1})^T A^T = (AA^{-1})^T$

$= I^T = I$

$\quad\;\; = A^T (A^{-1})^T$

$= (A^{-1} \cdot A)^T = I^T = I$



Strang Sections 2.6 – Elimination = Factorization: $A = LU$ and 2.7 – Transposes and Permutations

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Goal

$A = L \cdot U$

lower trainsular matrix

upper traingular matrix

How to calculate LU ?

Elimination Method

L. S. $\longrightarrow$ Upper Traingular Form

$\Downarrow$

using Elimination Matrix

$\Downarrow$

Gives you the L

Computing U – 2×2 case

We will start with a 2×2 matrix, then a 3×3 matrix, and then generalize to the $n \times n$ case.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \xrightarrow{\text{goal}} U = \begin{bmatrix} b & c \\ 0 & d \end{bmatrix}$$

If $a_{11} \neq 0$, then it is a pivot and we use it to eliminate a_{21}.

$$E_{21} \cdot A = U$$

$$\text{Row}(2) \leftarrow \text{Row}(2) + \left(-\frac{a_{21}}{a_{11}}\right) \cdot \text{Row}(1)$$

$$A = \underbrace{E_{21}^{-1}}_{L} \cdot U$$

$$A \to \underbrace{E_{21}}_{\begin{bmatrix} 1 & \\ -\frac{a_{21}}{a_{11}} & 1 \end{bmatrix}} \cdot A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} - \frac{a_{21}}{a_{11}} a_{12} \end{bmatrix}$$

$$E_{21}^{-1} = \begin{bmatrix} 1 & \\ \frac{a_{21}}{a_{11}} & 1 \end{bmatrix}$$

Computing U – 2×2 case

We will start with a 2×2 matrix, then a 3×3 matrix, and then generalize to the $n \times n$ case.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \xrightarrow{\text{goal}} U = \begin{bmatrix} b & c \\ 0 & d \end{bmatrix}$$

If $a_{11} \neq 0$, then it is a pivot and we use it to eliminate a_{21}.

lower traingular Matrix

upper traingular!!

$$E_{21}A = \begin{bmatrix} 1 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ 0 & \underbrace{a_{22} - \frac{a_{21}}{a_{11}} a_{12}}_{d} \end{bmatrix}$$

$\text{row}(2) \Leftarrow -\frac{a_{21}}{a_{11}} \cdot \text{row}(1) + \text{row}(2)$

$\underbrace{E_{21}}_{L^{-1}} A = U$

$A = \underbrace{E_{21}^{-1}}_{L} U$

Computing U – 2×2 case

We will start with a 2×2 matrix, then a 3×3 matrix, and then generalize to the $n \times n$ case.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \xrightarrow{\text{goal}} U = \begin{bmatrix} b & c \\ 0 & d \end{bmatrix}$$

If $a_{11} \neq 0$, then it is a pivot and we use it to eliminate a_{21}.

$$E_{21}A = \begin{bmatrix} 1 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ 0 & \underbrace{a_{22} - \frac{a_{21}}{a_{11}} a_{12}}_{d} \end{bmatrix}$$

$(E_{21})^{-1}$

Computing U – 2×2 case

We will start with a 2×2 matrix, then a 3×3 matrix, and then generalize to the $n \times n$ case.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \xrightarrow{\text{goal}} U = \begin{bmatrix} b & c \\ 0 & d \end{bmatrix}$$

If $a_{11} \neq 0$, then it is a pivot and we use it to eliminate a_{21}.

$$E_{21}A = \begin{bmatrix} 1 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ 0 & \underbrace{a_{22} - \frac{a_{21}}{a_{11}} a_{12}}_{d} \end{bmatrix}$$

If $a_{11} = 0$, but $a_{21} \neq 0$, we have to permute first. If both a_{11} and a_{21} are zero, then the matrix is already upper triangular.

Computing U – 3×3 case

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{\text{goal}} U = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ 0 & 0 & f \end{bmatrix}$$

If $a_{11} \neq 0$, then we make it first pivot and use it to eliminate a_{21} and a_{31}.

$$A \xrightarrow{1} E_{21} A \xrightarrow{2} E_{31} E_{21} A \xrightarrow{3} E_{32} E_{31} E_{21} A = U$$

$$E_{21} = \begin{bmatrix} 1 & & \\ -\frac{a_{21}}{a_{11}} & 1 & \\ & & 1 \end{bmatrix} \quad E_{31} = \begin{bmatrix} 1 & & \\ & 1 & \\ -\frac{a_{31}}{a_{11}} & & 1 \end{bmatrix} \quad E_{32} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -\frac{b_{32}}{b_{22}} & 1 \end{bmatrix}$$

$$E_{21} A = \begin{bmatrix} * & * & * \\ 0 & * & * \\ * & * & * \end{bmatrix} \quad E_{31} E_{21} A = \begin{bmatrix} * & * & * \\ 0 & b_{22} & * \\ 0 & b_{32} & * \end{bmatrix} \quad \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$$

$$\overbrace{E_{32} E_{31} E_{21}}^{L^{-1}} A = U \qquad A = L \cdot U, \qquad L = (E_{32} E_{31} E_{21})^{-1}.$$

Computing U – 3×3 case

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{\text{goal}} U = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ 0 & 0 & f \end{bmatrix}$$

If $a_{11} \neq 0$, then we make it first pivot and use it to eliminate a_{21} and a_{31}.

$$E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Computing U – 3×3 case

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{\text{goal}} U = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ 0 & 0 & f \end{bmatrix}$$

If $a_{11} \neq 0$, then we make it first pivot and use it to eliminate a_{21} and a_{31}.

$$E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{a_{31}}{a_{11}} & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ 0 & d & e \end{bmatrix}$$

Computing U – 3×3 case

If $a_{11} \neq 0$, then we make it first pivot and use it to eliminate a_{21} and a_{31}.

$$E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{a_{31}}{a_{11}} & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ 0 & d & e \end{bmatrix}$$

If $b \neq 0$, then we make it second pivot and use it to eliminate d.

$$\underbrace{E_{32}E_{31}E_{21}}_{(L^{-1}!)}A = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{d}{b} & 1 \end{bmatrix}}^{E_{32}} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ 0 & d & e \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ 0 & 0 & f \end{bmatrix}$$

Computing U – General Case

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \xrightarrow{\text{goal}} U = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & b & c & \dots & d \\ 0 & 0 & e & \dots & f \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & g \end{bmatrix}$$

1

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & b & * & \dots & * \\ 0 & * & * & \dots & * \\ \vdots & & & & \\ 0 & * & * & \dots & * \end{bmatrix}$$

step2

Computing U – General Case

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \xrightarrow{\text{goal}} U = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & b & c & \dots & d \\ 0 & 0 & e & \dots & f \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & g \end{bmatrix}$$

n-1 1 2 1 n-2

$$a_{11} \text{ pivot} \xrightarrow{1} E_{21}A \xrightarrow{2} E_{31}E_{21}A \rightarrow E_{41}E_{31}E_{21}A \rightarrow \underbrace{\overset{n-1}{E_{n1}} \dots E_{41}E_{31}E_{21}A}_{B}$$

$$b \text{ pivot} \xrightarrow{1} E_{32}B \xrightarrow{2} E_{42}E_{32}B \rightarrow E_{52}E_{42}E_{32}B \rightarrow \underbrace{\overset{n-2}{E_{n2}} \dots E_{52}E_{42}E_{32}B}_{C}$$

$$e \text{ pivot} \rightarrow E_{43}C \rightarrow E_{53}E_{43}C \rightarrow E_{63}E_{53}E_{43}C \rightarrow E_{n3} \dots E_{63}E_{53}E_{43}C$$

$$\vdots$$

$$\underbrace{E_{n,n-1}\, E_{n,n-2}\, E_{n-1,n-2} \cdots E_{n1} \cdots E_{41}E_{31}E_{21}}_{L^{-1}}\, A = U$$

note that we're assuming we can find a pivot without having to use permutations

$$A = L \cdot U$$

Computing U – General Case

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \xrightarrow{\text{goal}} U = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & b & c & \cdots & d \\ 0 & 0 & e & \cdots & f \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & g \end{bmatrix}$$

Step1 : (n-1) Elimination Metric : $E_{n1} \cdots E_{21}$!! order. operate first

a_{11} pivot $\longrightarrow E_{21}A \to E_{31}E_{21}A \to E_{41}E_{31}E_{21}A \to \underbrace{E_{n1}\ldots E_{41}E_{31}E_{21}A}_{B}$

Step 2: (n-2) Elimination Matrix

b pivot $\longrightarrow E_{32}B \to E_{42}E_{32}B \to E_{52}E_{42}E_{32}B \to \underbrace{E_{n2}\ldots E_{52}E_{42}E_{32}B}_{C}$

Step 3: (n-3) Elimation Matrix

e pivot $\longrightarrow E_{43}C \to E_{53}E_{43}C \to E_{63}E_{53}E_{43}C \to E_{n3}\ldots E_{63}E_{53}E_{43}C$

$\vdots$

note that we're assuming we can find a pivot without having to use permutations

Computing L

2×2 case: If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then $U = E_{21}A$.

$$\implies A = \underbrace{E_{21}^{-1}}_{L} U$$

3×3 case: If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $U = E_{32}E_{31}E_{21}A$.

$$\implies A = (E_{32}E_{31}E_{21})^{-1}U$$

$$= \underbrace{E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}}_{L} U$$

Goal

$$A = L\,D\,U$$

$$\begin{pmatrix} d_{11} & & \\ & \ddots & \\ & & d_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} d_{11}x_1 \\ \vdots \\ d_{nn}x_n \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \qquad D = \begin{pmatrix} d_{11} & & \\ & \ddots & \\ & & d_{nn} \end{pmatrix} \qquad U = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

Lower Triangular but diag are 1

diagonal Matrix

Upper Triangular but diag are 1

Find LU decomposition first.

$$\begin{bmatrix} 2 & 1 & \\ 1 & 2 & 1 \\ & 1 & 2 \end{bmatrix} = \overset{L}{\begin{bmatrix} 1 & & \\ 1/2 & 1 & \\ & 1/3 & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 2 & 1 & 0 \\ & 3/2 & 1 \\ & & 4/3 \end{bmatrix}} = L \underbrace{\begin{bmatrix} 2 & & \\ & 3/2 & \\ & & 4/3 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 1/2 & \\ & 1 & 2/3 \\ & & 1 \end{bmatrix}}_{U}$$

LDU decomposition!

Questions?