Lecture 5

Matrix Operations and Inverse Matrix

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Matrix Operations

Matrix multiplication

$$\vec{r}_i^{\,T} \in \mathbb{R}^n \qquad \vec{v}_j \in \mathbb{R}^n$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ik} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mk} & \cdots & a_{mn} \end{pmatrix} \text{ith row} \cdot \begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{k1} & \cdots & b_{kj} & \cdots & b_{kp} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mp} \end{pmatrix}$$

$\mathbb{R}^{m\times n}$ jth column $\mathbb{R}^{n\times p}$ ij entry $\mathbb{R}^{m\times p}$

$$C_{ij} = \vec{r}_i^{\,T} v_j = a_{i1} \cdot b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}$$

Question : *A* is a matrix, write a matrix-vector multiplication as the first column

$A = [\, \vec{v}_1 \ \ \vec{v}_2 \ \cdots \ \vec{v}_n \,]$. find a vector $\underset{\mathbb{R}^n}{\vec{x}}$ such that $A x = \underbrace{\vec{v}_1}_{\mathbb{R}^m}$

$A \in \mathbb{R}^{m\times n}$

$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $A\vec{x} = x_1 \vec{v}_1 + x_2 \vec{v}_2 + \cdots + x_n \vec{v}_n$

I want $Ax = \vec{v}_1$. then $x_1 = 1$. $x_2 = \cdots = x_n = 0$

$\Rightarrow \quad x = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Follow Up Question,

Matrix $I_n = \begin{pmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix}$. what is AI_n $\quad \mathbb{R}^{m\times n} \ \mathbb{R}^{n\times n} \rightarrow \mathbb{R}^{m\times n}$

① first Column vector of AI_n

$A[\, v_1' \ v_2' \cdots v_n' \,] = [\, Av_1' \ \ Av_2' \ \ \cdots \ \ Av_n' \,]$

Column vectors

$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

$AI_n = A \Leftarrow [\, v_1 \quad v_2 \quad \cdots \quad v_n \,]$

Matrix multiplication

$$\underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ik} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mk} & \cdots & a_{mn} \end{pmatrix}}_{\mathbb{R}^{m\times n}} \cdot \underbrace{\begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{k1} & \cdots & b_{kj} & \cdots & b_{kp} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{pmatrix}}_{\mathbb{R}^{n\times p}} = \underbrace{\begin{pmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mp} \end{pmatrix}}_{\mathbb{R}^{m\times p}}$$

ith row ($\mathbb{R}^n$) jth column ($\mathbb{R}^n$) ij entry

$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$

Question : *A* is a matrix, write a matrix-vector multiplication as the first column

$A = [\vec{v}_1 \ \cdots \ \vec{v}_n]$ ($\mathbb{R}^{m\times n}$), what is the vector $x \in \mathbb{R}^n$ such that $Ax = v_1$

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$A\vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n$

– I want $A\vec{x} = \vec{v}_1$ $x_1 = 1$ $x_2 = \cdots x_n = 0$ $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Similarly if $\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ what is $A\vec{x} = \vec{v}_2$ second column of A

$\vec{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ $A\vec{x}$ Third column .....

Follow up Question

What is $AI_n = A$

$A\begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & & 1 \end{bmatrix}$

$= \Big[\underbrace{A\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\text{the first col of } A}, \underbrace{A\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}}_{\text{the second col of } A} \cdots \underbrace{A\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}}_{\text{the last col of } A}\Big]$

$= A$

Quiz: $I_m A = A$

$I_m = \begin{bmatrix} (1, 0, \cdots 0) \\ (0, 1, \cdots, 0) \\ \vdots \\ (0, 0, \cdots, 1) \end{bmatrix}$ A → first row of A

Matrix multiplication

$$\begin{pmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ik} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mk} & \cdots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{k1} & \cdots & b_{kj} & \cdots & b_{kp} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mp} \end{pmatrix}$$

ith row

jth column

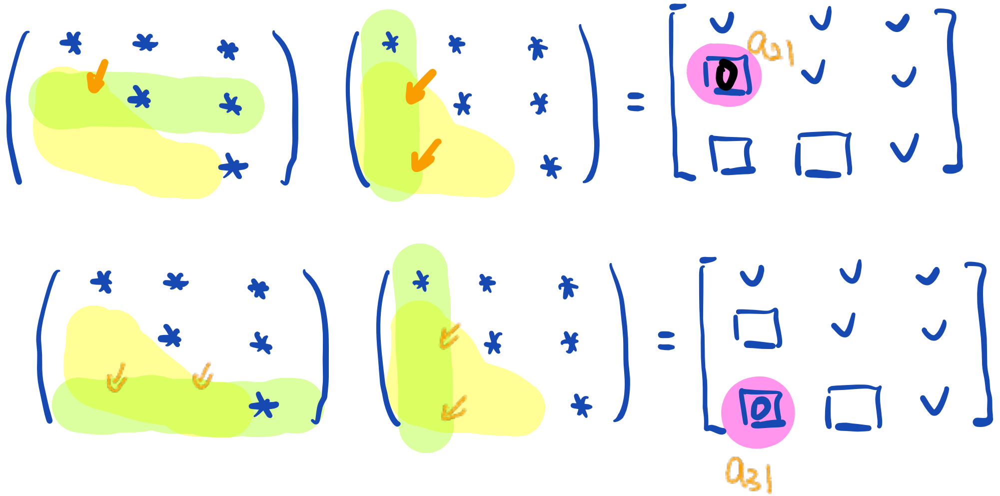
ij entry

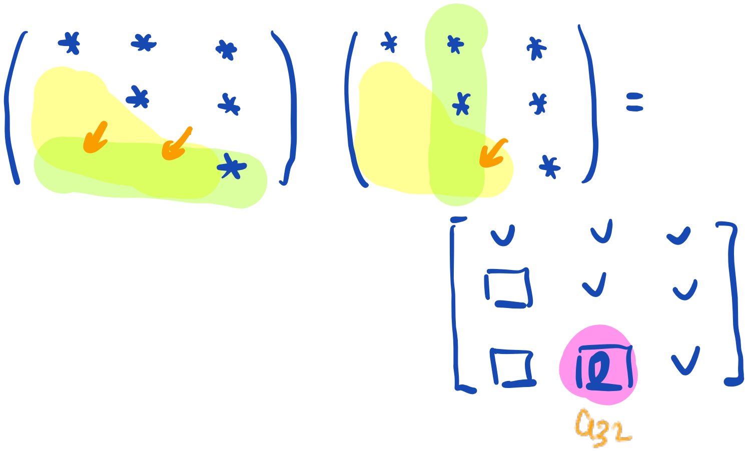
Lower Triangular Matrix × Lower Triangular Matrix → Lower Triangular Matrix

Question : *A* is a matrix, write a matrix-vector multiplication as the first column

Is upper triangular matrix times a upper triangular matrix still upper triangular

Yes!





Diagonal Matrix

Let $D = \begin{bmatrix} d_{11} & & & \\ & d_{22} & & \\ & & \ddots & \\ & & & d_{nn} \end{bmatrix}$ be an $n \times n$ diagonal matrix,

Square Matrix

What is Dx?

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$D \cdot x = \begin{bmatrix} d_{11} & & & \\ & d_{22} & & \\ & & \ddots & \\ & & & d_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_{11} \cdot x_1 \\ d_{22} \cdot x_2 \\ \vdots \\ d_{nn} \cdot x_n \end{bmatrix}$$

(☆).

Inverse Matrix

Defn I

linear System

- Inverse Matrix $Ax = b \Leftrightarrow x = A^{-1}b$ is the only solution for all vector b

Suppose A is an $n \times n$ matrix (square matrix), then A is invertible if there exists a matrix A^{-1} such that

check.

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I.$$

← another definition.

We can only talk about an inverse of a square matrix, but not all square matrices are invertible. We will discuss such restrictions in future lectures.

$A^{-1}A$ is I_n $\Leftarrow$.

$A^{-1}A \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ = using defn! $\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ← Solve the linear System $Ax = A\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

the first Column of $A^{-1}A$

the only Solution is $A^{-1}\left[A\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right]$

$A^{-1}A \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ = using defn I $\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

there is a solution $\begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix}$))

the second Column of $A^{-1}A$

Example

Example

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

I claim $B = A^{-1}$. Check:

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This also means

$$\begin{cases} 2x + y = a \\ x + y = b \end{cases}$$

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

The solution is

$$\begin{cases} x = a - b \\ y = -a + 2b \end{cases}$$

$$B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = B \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} a \\ b \end{pmatrix}$$

Inverse of a Diagonal Matrix

$D \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \to \begin{pmatrix} d_{11} \; x_1 \\ \vdots \\ d_{nn} \; x_n \end{pmatrix}$

Let $D = \begin{bmatrix} d_{11} & & & \\ & d_{22} & & \\ & & \ddots & \\ & & & d_{nn} \end{bmatrix}$ be an $n \times n$ diagonal matrix, then

$\begin{bmatrix} 1/d_{11} & & & \\ & 1/d_{22} & & \\ & & \ddots & \\ & & & 1/d_{nn} \end{bmatrix}$

$$D^{-1} = \begin{bmatrix} 1/d_{11} & & & \\ & 1/d_{22} & & \\ & & \ddots & \\ & & & 1/d_{nn} \end{bmatrix}$$ provided that $d_{ii} \neq 0$.

Recall

Suppose we are given a system of m equations in n unknowns:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

This system can be written in matrix form as:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\xrightarrow{\text{in augmented form}} \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

$$[A|b] \to [I|\underbrace{\vec{x}}_{Ax = b}]$$

Two Operations

- Linear combine two rows 'ellimination'

  Replace Row (i) with $\underbrace{*\cdot(j) + (i)}_{\text{L.C. of (j) and (i)}}$

  Ellimination Matrix

  $$[A \mid b] \xrightarrow{\text{Replace Row (i) with } *\cdot(j) + (i)} [\underbrace{E_{ij} A}_{\text{Matrix} \times \text{Matrix}} \mid E_{ij} b]$$

- Permutation

  Switch Two Rows (i) and (j)

  Permutation Matrix

  $$[A \mid b] \xrightarrow{\text{Switch Two Rows (i) and (j)}} [P_{ij} A \mid P_{ij} b]$$

Permutation Matrices

Recall

$$I = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \xrightarrow{I\,\vec{x}} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

Permutation Matrices

$$P_{ij} = \begin{array}{c} \left[\begin{array}{cccccccc} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{array}\right] \end{array} \begin{array}{l} \\ \\ \\ \leftarrow i\text{-th row} \\ \\ \leftarrow j\text{-th row} \\ \\ \\ \end{array}$$

a_{ii} a_{ij}

↑ i-th Column   ↑ j-th Column

$a_{ii} = a_{jj} = 0$ $a_{ij} = a_{ji} = 1$

$$P_{ij} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix}$$

$P_{ij} A \rightsquigarrow$ switch i-th row and j-th row of A

Permutation Matrices

$$P_{ij} = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots \\ \color{blue}0 & \color{blue}0 & \color{blue}\dots & \color{blue}0 & \color{blue}\dots & \color{blue}1 & \color{blue}\dots & \color{blue}0 \\ \vdots \\ \color{red}0 & \color{red}0 & \color{red}\dots & \color{red}1 & \color{red}\dots & \color{red}0 & \color{red}\dots & \color{red}0 \\ \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

$$P_{ij} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \color{blue}x_j \\ \vdots \\ \color{red}x_i \\ \vdots \\ x_n \end{bmatrix}$$

$$P_{31} = \begin{bmatrix} \color{blue}0 & \color{blue}0 & \color{blue}1 & \color{blue}\dots & \color{blue}0 \\ 0 & 1 & 0 & \dots & 0 \\ \color{red}1 & \color{red}0 & \color{red}0 & \color{red}\dots & \color{red}0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \xrightarrow{P_{31}\,\vec{x}} \begin{bmatrix} \color{blue}0 & \color{blue}0 & \color{blue}1 & \color{blue}\dots & \color{blue}0 \\ 0 & 1 & 0 & \dots & 0 \\ \color{red}1 & \color{red}0 & \color{red}0 & \color{red}\dots & \color{red}0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_3 \\ x_2 \\ x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Elimination Matrices

Elimination Matrices

$$E_{ji} = \begin{array}{cc}
\begin{array}{cccccccc} & & & \text{Col } i & & \text{Col } j & & \end{array} & \\
\begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \ddots & & & & & \\ 0 & 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \ddots & & & \\ 0 & 0 & \cdots & \overset{a_{ji} = *}{\star} & \cdots & 1 & \cdots & 0 \\ \vdots & & & & & & \ddots & \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} & \begin{array}{l} \\ \\ \\ \text{Row } i \\ \\ \text{Row } j \\ \\ \\ \end{array}
\end{array} \Rightarrow$$

change my Row (j)

with $(*) \cdot$ Row (i)

$+$ Row (j)

Elimination Matrices

$$E_{ji} = \begin{array}{cc} \begin{array}{cccccccc} & & & \text{Col } i & & \text{Col } j & & \end{array} & \\ \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \ddots & & & & & \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & & & \ddots & & & \\ 0 & 0 & \dots & \star & \dots & 1 & \dots & 0 \\ \vdots & & & & & & \ddots & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix} & \begin{array}{l} \\ \\ \\ \text{Row } i \\ \\ \text{Row } j \\ \\ \\ \end{array} \end{array}$$

$$E_{ji} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j + (\star \cdot x_i) \\ \vdots \\ x_n \end{bmatrix}$$

Elimination Matrices

What does the matrix $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ do to the vector $\vec{x} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$ when it acts on it?

a_{21}

$E_{21}\,x$ Replace Row (2) with $-2 *$ Row (1) + Row (2)

$$E_{21}\vec{x} = \begin{bmatrix} 2 \\ 8 + (-2) \times 2 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}$$

Elimination Matrices

What does the matrix $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ do to the vector $\vec{x} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}$ when it acts on it?

a_{31}

Replace Row (3) with Row (1) + Row (3)

$$\begin{bmatrix} 2 \\ 4 \\ 10+2=12 \end{bmatrix}$$

Solving Linear Systems

Elimination

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

$$[A \mid b] \longrightarrow [I \mid \vec{x}]$$

Column by Column

Elimination – Summary of the previous example

$$\left(\begin{array}{ccc|c} \color{red}{1} & 2 & 3 & 6 \\ \boxed{2} & -3 & 2 & 14 \\ \boxed{3} & 1 & -1 & -2 \end{array}\right) \xrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ \color{red}{0} & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 - 3R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ \color{red}{0} & -5 & -10 & -20 \end{array}\right)$$

First Glum

We want these to be zero.
So we subract multiples of the first row.

$$\left(\begin{array}{ccc|c} 1 & \boxed{2} & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & \boxed{-5} & -10 & -20 \end{array}\right) \xrightarrow{R_2 \longleftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 \div -5} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & \color{red}{1} & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

$$\xrightarrow{R_1 = R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & \color{red}{0} & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 + 7R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & \color{red}{0} & 10 & 30 \end{array}\right)$$

Second Glum

We want these to be zero.

It would be nice if this were a 1.
We could divide by -7, but that
would produce ugly fractions.

Let's swap the last two rows first.

Elimination

$$\left(\begin{array}{ccc|c} 1 & 0 & \boxed{-1} & -2 \\ 0 & 1 & \boxed{2} & 4 \\ 0 & 0 & 10 & 30 \end{array}\right) \overset{R_3 = R_3 \div 10}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

We want these to be zero.

Let's make this a 1 first.

$$\overset{R_1 = R_1 + R_3}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Third Column

$$\overset{R_2 = R_2 - 2R_3}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\overset{\text{translates into}}{\rightsquigarrow} \begin{aligned} x \quad &= 1 \\ y \quad &= -2 \\ z &= 3 \end{aligned}$$

Success!

Check:

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned} \overset{\text{substitute solution}}{\rightsquigarrow} \begin{aligned} 1 + 2 \cdot (-2) + 3 \cdot 3 &= 6 \\ 2 \cdot 1 - 3 \cdot (-2) + 2 \cdot 3 &= 14 \\ 3 \cdot 1 + (-2) - 3 &= -2 \end{aligned}$$

✔

The idea of Inverse Matrices

Consider the following system:

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 2 \\ 4x_1 + 9x_2 - 3x_3 &= 8 \\ -2x_1 - 3x_2 + 7x_3 &= 10 \end{aligned}$$

Our goal is to find x_1, x_2, and x_3. In matrix form, this system is:

$$\underbrace{\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

$$A \vec{x} = \vec{b}$$

$$\xrightarrow{\text{idea}} \quad \vec{x} = A^{-1} \vec{b}$$

$[A \mid \vec{b}] \leftarrow [A \mid I]$

$[I \mid A^{-1}\vec{b}] \leftarrow [I \mid A^{-1} I]$

$= [I \mid A^{-1}$

The idea of Inverse Matrices

$$\underbrace{\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

$$A \qquad \vec{x} = \vec{b}$$

$$\Longleftrightarrow \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

$$A \qquad \vec{x} = I \qquad \vec{b}$$

$$\xrightarrow{\text{idea}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

$$I \qquad \vec{x} = A^{-1} \qquad \vec{b}$$

The idea of Inverse Matrices

$$\left[\begin{array}{c|c} A & I \end{array}\right]$$

$$\Downarrow \textbf{elimination}$$

$$\left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$$

Two operation

Column by column.

Example

Example: Find the inverse of $A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 4 & 9 & -3 & 0 & 1 & 0 \\ -2 & -3 & 7 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 - 2R_1 \\ R_3 + R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 5 & 1 & 0 & 1 \end{array}\right] R_3 - R_2$$

$$\left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 4 & 3 & -1 & 1 \end{array}\right] R_3 \div 4$$

$$\left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3/4 & -1/4 & 1/4 \end{array}\right] R_1 - 4R_2$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & -6 & 9 & -4 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3/4 & -1/4 & 1/4 \end{array}\right] \begin{array}{l} R_1 + 6R_3 \\ R_2 - R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 27/2 & -11/2 & 3/2 \\ 0 & 1 & 0 & -11/4 & 5/4 & -1/4 \\ 0 & 0 & 1 & 3/4 & -1/4 & 1/4 \end{array}\right] R_1 \tfrac{1}{2}$$

Example

$$A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 27/4 & -11/4 & 3/4 \\ 0 & 1 & 0 & -11/4 & 5/4 & -1/4 \\ 0 & 0 & 1 & 3/4 & -1/4 & 1/4 \end{array}\right]$$

Inver Matrix

Solution

Linear System

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = b_1 \\ 4x_1 + 9x_2 - 3x_3 = b_2 \\ -2x_1 - 3x_2 + 7x_3 = b_3 \end{cases}$$

$$A\vec{x} = \vec{b}$$

$$\begin{cases} b_1 = 27/4\, x_1 - 11/4\, x_2 + 3/4\, x_3 \\ b_2 = -11/4\, x_1 + 5/4\, x_2 - 1/4\, x_3 \\ b_3 = 3/4\, x_1 - 1/4\, x_2 + 1/4\, x_3 \end{cases}$$

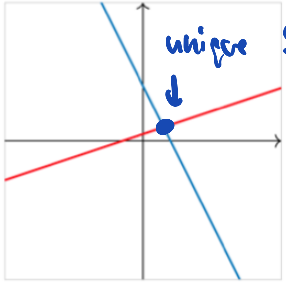
$$\vec{b} = A^{-1}\vec{x}$$

Three Cases

Not all the Matrix have inverse.

$x - 3y = -3$

$2x + y = 8$



unique Solution

$$\left(\begin{array}{cc|c} 1 & -3 & -3 \\ 2 & 1 & 8 \end{array}\right) \quad R2 - 2*R1$$

$$\rightarrow \left(\begin{array}{cc|c} 1 & -3 & -3 \\ 0 & 7 & 14 \end{array}\right) \begin{array}{l} R1' \\ R2' \end{array}$$

← upper Triangular Form

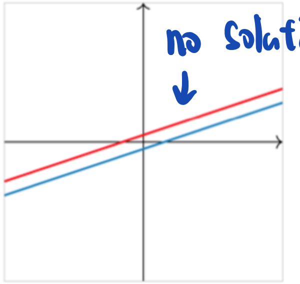
Two pivot!!

using $R2' \rightarrow x_2 = 2$

using $x_2 = 2$ and $R1' \rightarrow x_1$

$x - 3y = -3$

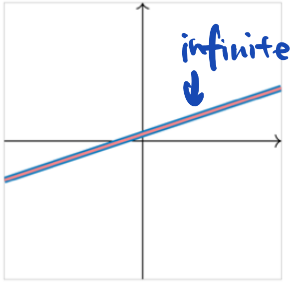
$x - 3y = 3$



no Solution

$x - 3y = -3$

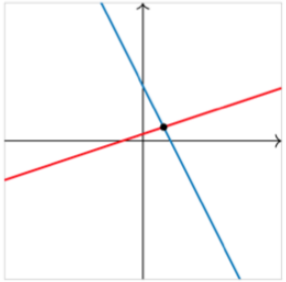
$2x - 6y = -6$



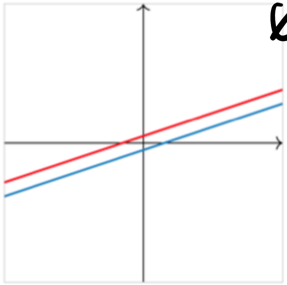
infinite Solution.

Three Cases

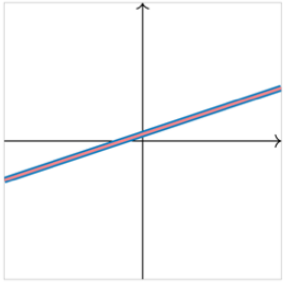
$x - 3y = -3$
$2x + y = 8$



$x - 3y = -3$
$x - 3y = 3$



$x - 3y = -3$
$2x - 6y = -6$



$$\left(\begin{array}{cc|c} 1 & -3 & -3 \\ 1 & -3 & 3 \end{array}\right) \quad R2 - R1$$

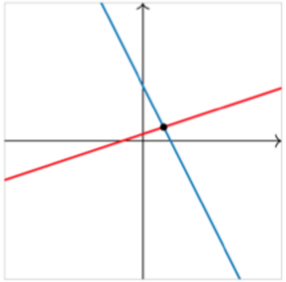
$$\left(\begin{array}{cc|c} 1 & -3 & -3 \\ 0 & \boxed{0} & 6 \end{array}\right)$$

only one pivot

it should be zero ! No solution

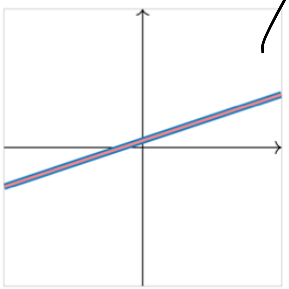
Three Cases

$x - 3y = -3$
$2x + y = 8$



$x - 3y = -3$
$x - 3y = 3$

$x - 3y = -3$
$2x - 6y = -6$



$$\left(\begin{array}{cc|c} 1 & -3 & -3 \\ 2 & -6 & -6 \end{array}\right) \quad R2 - 2R1$$

$$\left(\begin{array}{cc|c} 1 & -3 & -3 \\ 0 & 0 & 0 \end{array}\right)$$

R1' 1 pivot

x2 can be any value

Using R1' to solve x_1

$$\begin{cases} x_2 = x_2 \\ x_1 = -3 + 3x_2 \end{cases}$$

Example

(1) Using Elimination to change Linear System to Upper Trianglar Form

$$\left(\begin{array}{cc|c} 1 & * & * \\ 0 & * & * \end{array}\right)$$

→ using this to determin x_1

→ using this to determine x_2

Can be zero.

$$\left(\begin{array}{cc|c} 0 & 0 & 0 \end{array}\right)$$ → means $0 \cdot x_1 + 0\, x_2 = 0$ ⇒ x_2 can be any things

Note on Infinite Solutions

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \to x_1 \\ \to x_2 \\ \leftarrow x_3 \end{array}$$

$x_1 + 3x_3 = 4$ $\qquad x_1 = 4 - 3x_3$

$x_2 - 2x_3 = 1$ $\qquad x_2 = 2x_3 + 1$

$0x_1 + 0x_2 + 0x_3 = 0$

x_3 can be any thing

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + x_3 \cdot \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} \leftarrow \text{line}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \to x_1 \\ \to x_2 \\ \to x_3 \\ \leftarrow x_4 \end{array}$$

$x_1 = 4 - x_3 = 4 - (2x_4 + 1) = 3 - 2x_4$

x_2 can be anything

$x_3 = 2x_4 + 1$

x_4 can be anything

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \leftarrow \text{plane}$$

Note on Infinite Solutions

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & \underline{0} & 0 \end{array}\right]$$

$\leftarrow x_1 = 4 - 3x_3$ $\rightarrow x_1 = 4 - 3 \times (\frac{1}{2} - x_2) = 3x_2 + \frac{5}{2}$

$\leftarrow x_2 = 1 + 2x_3$ $\rightarrow x_3 = \frac{1}{2} - x_2$

$\leftarrow x_3$ can be any value.

if it's not 0, no solution

Equal to $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$ ← This is all my solution!

Example

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\rightarrow x_1 = 4 - x_2 - x_3 - 0 \cdot x_4 = 4 - x_2 - (1 - 2x_4) = 3 - x_2 + 2x_4$

4 unknowns, 3 Eq.

$\rightarrow x_2$ can be any vale

$\rightarrow x_3 = 1 - 2x_4$

$\rightarrow x_4$ can be any vale

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ 0 \\ -2 \\ 1 \end{pmatrix}$ ← This is all solutions!

Block Matrix*

Block Matrices

4 by 6 matrix
2 by 2 blocks

$$A = \left[\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right] = \begin{bmatrix} I & I & I \\ I & I & I \end{bmatrix}.$$

Block multiplication If the cuts between columns of A match the cuts between rows of B, then block multiplication of AB is allowed:

$$\begin{array}{cc} & \begin{matrix} n_1 & \quad n_2 \end{matrix} \\ \begin{matrix} m_1 \\ m_2 \end{matrix} & \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \end{array} \begin{array}{cc} \begin{matrix} k_1 & \quad \end{matrix} & \\ \begin{bmatrix} B_{11} & \cdots \\ B_{21} & \cdots \end{bmatrix} & \begin{matrix} n_1 \\ n_2 \end{matrix} \end{array} = \begin{array}{cc} \begin{matrix} m_1 \\ m_2 \end{matrix} & \begin{bmatrix} \overset{n_1}{A_{11}}\overset{k_1}{B_{11}}{}^{n_1} + \overset{n_2}{A_{12}}\overset{k_1}{B_{21}}{}^{n_2} & \cdots \\ A_{21}B_{11} + A_{22}B_{21} & \cdots \end{bmatrix} \end{array}. \qquad (1)$$

Example 3 **(Important special case)** Let the blocks of A be its n columns. Let the blocks of B be its n rows. Then block multiplication AB adds up ***columns times rows***:

$$\begin{array}{l}\textbf{Columns}\\ \textbf{times}\\ \textbf{rows}\end{array} \qquad \begin{bmatrix} | & & | \\ \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_n \\ | & & | \end{bmatrix} \begin{bmatrix} - & \boldsymbol{b}_1 & - \\ & \vdots & \\ - & \boldsymbol{b}_n & - \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_1\boldsymbol{b}_1 + \cdots + \boldsymbol{a}_n\boldsymbol{b}_n \end{bmatrix}. \qquad (2)$$

$$\boxed{AB = \begin{pmatrix} - r_1 - \\ \vdots \\ - r_m - \end{pmatrix} \begin{pmatrix} | & & | \\ c_1 & \cdots & c_p \\ | & & | \end{pmatrix} = \begin{pmatrix} r_1c_1 & r_1c_2 & \cdots & r_1c_p \\ r_2c_1 & r_2c_2 & \cdots & r_2c_p \\ \vdots & \vdots & & \vdots \\ r_mc_1 & r_mc_2 & \cdots & r_mc_p \end{pmatrix}}$$

Elimination by Block

One at a time $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$.

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

Block elimination

$$\left[\begin{array}{c|c} I & \mathbf{0} \\ \hline -CA^{-1} & I \end{array}\right]\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline \mathbf{0} & D - CA^{-1}B \end{array}\right].$$