NYU

Linear Algebra

Lecture 5

Matrix Operations and Inverse Matrix

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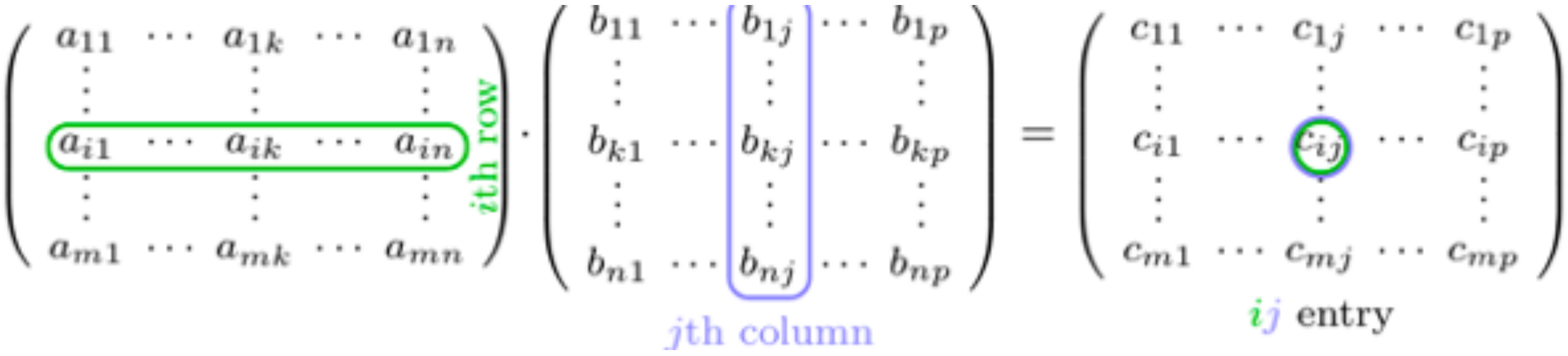
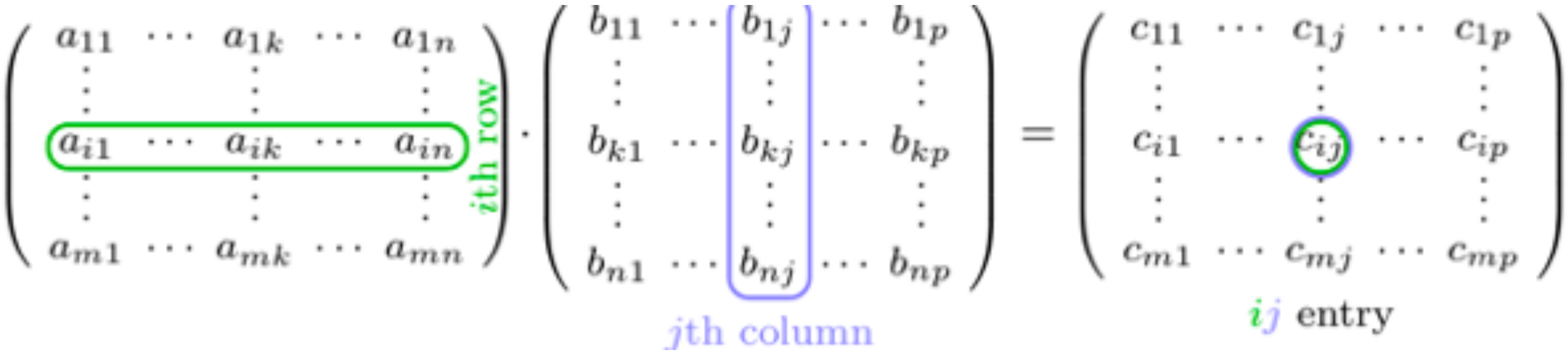
Matrix Operations

Matrix multiplication



Question : *A* is a matrix, write a matrix-vector multiplication as the first column

Matrix multiplication



Question : *A* is a matrix, write a matrix-vector multiplication as the first column

Is upper triangular matrix times a upper triangular matrix still upper triangular

Diagonal Matrix

Let $D = \begin{bmatrix} d_{11} & & & \\ & d_{22} & & \\ & & \ddots & \\ & & & d_{nn} \end{bmatrix}$ be an $n \times n$ diagonal matrix,

What is Dx?

Inverse Matrix

- Inverse Matrix $Ax = b \Leftrightarrow x = A^{-1}b$

Suppose A is an $n \times n$ matrix (square matrix), then A is invertible if there exists a matrix A^{-1} such that

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I.$$

We can only talk about an inverse of a square matrix, but not all square matrices are invertible. We will discuss such restrictions in future lectures.

Example

Example
$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

I claim $B = A^{-1}$. Check:

Inverse of a Diagonal Matrix

Let $D = \begin{bmatrix} d_{11} & & & \\ & d_{22} & & \\ & & \ddots & \\ & & & d_{nn} \end{bmatrix}$ be an $n \times n$ diagonal matrix, then

$D^{-1} = \begin{bmatrix} 1/d_{11} & & & \\ & 1/d_{22} & & \\ & & \ddots & \\ & & & 1/d_{nn} \end{bmatrix}$ provided that $d_{ii} \neq 0$.

Recall

Suppose we are given a system of m equations in n unknowns:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

This system can be written in matrix form as:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

in augmented form →

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

Two Operations

- Linear combine two rows

- Permutation

Permutation Matrices

Recall

$$I = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \xrightarrow{I\,\vec{x}} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

Permutation Matrices

$$P_{ij} = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

$$P_{ij} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix}$$

Permutation Matrices

$$P_{ij} = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

$$P_{ij} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix}$$

$$P_{31} = \begin{bmatrix} 0 & 0 & 1 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \xrightarrow{P_{31}\,\vec{x}} \begin{bmatrix} 0 & 0 & 1 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

Elimination Matrices

Elimination Matrices

Col i Col j

$$E_{ji} = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \ddots & & & & & \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & & & \ddots & & & \\ 0 & 0 & \dots & \star & \dots & 1 & \dots & 0 \\ \vdots & & & & & & \ddots & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ \text{Row } i \\ \\ \text{Row } j \\ \\ \\ \end{matrix}$$

Elimination Matrices

Col i Col j

$$
E_{ji} = \begin{bmatrix}
1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\
0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\
\vdots & & \ddots & & & & & \\
0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\
\vdots & & & & \ddots & & & \\
0 & 0 & \dots & \star & \dots & 1 & \dots & 0 \\
\vdots & & & & & & \ddots & \\
0 & 0 & \dots & 0 & \dots & 0 & \dots & 1
\end{bmatrix}
$$

Row i

Row j

$$
E_{ji} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j + (\star \cdot x_i) \\ \vdots \\ x_n \end{bmatrix}
$$

Elimination Matrices

What does the matrix $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ do to the vector $\vec{x} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$ when it acts on it?

Elimination Matrices

What does the matrix $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ do to the vector $\vec{x} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}$ when it acts on it?

Solving Linear Systems

Elimination

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

Elimination – Summary of the previous example

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right) \xrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 - 3R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right)$$

We want these to be zero.
So we subract multiples of the first row.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right) \xrightarrow{R_2 \longleftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 \div -5} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

$$\xrightarrow{R_1 = R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 + 7R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right)$$

We want these to be zero.

It would be nice if this were a 1.
We could divide by -7, but that would produce ugly fractions.

Let's swap the last two rows first.

Elimination

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right) \xrightarrow{R_3 = R_3 \div 10} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

We want these to be zero.

Let's make this a 1 first.

$$\xrightarrow{R_1 = R_1 + R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 - 2R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{\text{translates into}} \begin{aligned} x \quad\quad &= 1 \\ y \quad &= -2 \\ z &= 3 \end{aligned}$$

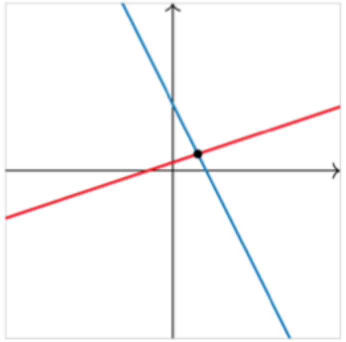
Success!

Check:

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned} \xrightarrow{\text{substitute solution}} \begin{aligned} 1 + 2 \cdot (-2) + 3 \cdot 3 &= 6 \\ 2 \cdot 1 - 3 \cdot (-2) + 2 \cdot 3 &= 14 \\ 3 \cdot 1 + (-2) - 3 &= -2 \end{aligned} ✔$$

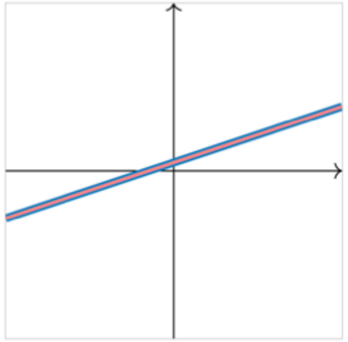
Three Cases

$$x - 3y = -3$$
$$2x + y = 8$$



$$x - 3y = -3$$
$$x - 3y = 3$$

$$x - 3y = -3$$
$$2x - 6y = -6$$



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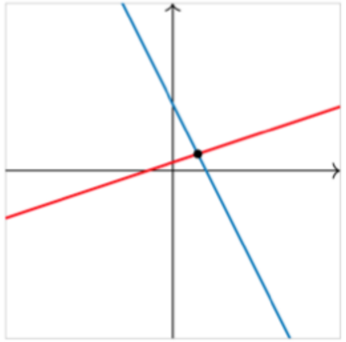

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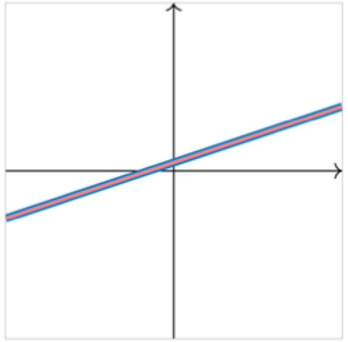

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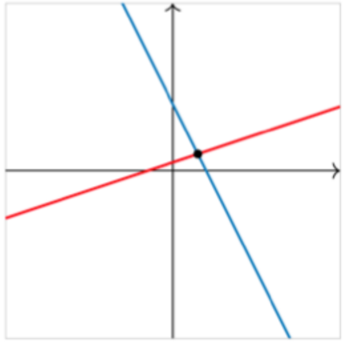

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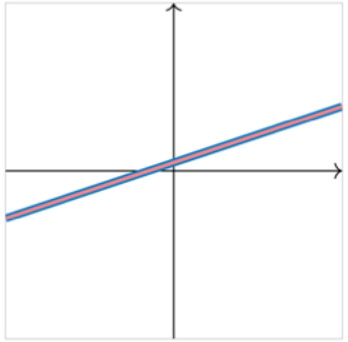

The idea of Inverse Matrices

Consider the following system:

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 2 \\ 4x_1 + 9x_2 - 3x_3 &= 8 \\ -2x_1 - 3x_2 + 7x_3 &= 10 \end{aligned}$$

Our goal is to find x_1, x_2, and x_3. In matrix form, this system is:

$$\underbrace{\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

$$A \quad \vec{x} = \vec{b}$$

idea →

$$\vec{x} = A^{-1} \vec{b}$$

The idea of Inverse Matrices

$$\underbrace{\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

$$A \qquad \vec{x} = \vec{b}$$

$$\iff \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

$$A \qquad \vec{x} = I \qquad \vec{b}$$

idea →

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

$$I \qquad \vec{x} = A^{-1} \qquad \vec{b}$$

The idea of Inverse Matrices

$$\left[\begin{array}{c|c} A & I \end{array}\right]$$

$$\Downarrow \textbf{ elimination}$$

$$\left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$$

Example

Example: Find the inverse of $A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}$.

Example

Example

Example

Note on Infinite Solutions

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Block Matrix*

Block Matrices

4 by 6 matrix
2 by 2 blocks

$$A = \left[\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right] = \begin{bmatrix} I & I & I \\ I & I & I \end{bmatrix}.$$

Block multiplication If the cuts between columns of A match the cuts between rows of B, then block multiplication of AB is allowed:

$$\begin{array}{c} \\ m_1 \\ m_2 \end{array}\overset{\quad n_1 \qquad n_2}{\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}} \overset{k_1}{\begin{bmatrix} B_{11} & \cdots \\ B_{21} & \cdots \end{bmatrix}}\begin{array}{c} n_1 \\ n_2 \end{array} = \begin{array}{c} m_1 \\ m_2 \end{array}\begin{bmatrix} A_{11}^{\;n_1} B_{11}^{\;k_1}\,{}_{n_1} + A_{12}^{\;n_2} B_{21}^{\;k_1}\,{}_{n_2} & \cdots \\ A_{21} B_{11} + A_{22} B_{21} & \cdots \end{bmatrix}. \tag{1}$$

Example 3 **(Important special case)** Let the blocks of A be its n columns. Let the blocks of B be its n rows. Then block multiplication AB adds up ***columns times rows***:

Columns times rows

$$\begin{bmatrix} | & & | \\ \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_n \\ | & & | \end{bmatrix} \begin{bmatrix} - & \boldsymbol{b}_1 & - \\ & \vdots & \\ - & \boldsymbol{b}_n & - \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_1\boldsymbol{b}_1 + \cdots + \boldsymbol{a}_n\boldsymbol{b}_n \end{bmatrix}. \quad (2)$$

$$AB = \begin{pmatrix} - r_1 - \\ \vdots \\ - r_m - \end{pmatrix} \begin{pmatrix} | & & | \\ c_1 & \cdots & c_p \\ | & & | \end{pmatrix} = \begin{pmatrix} r_1c_1 & r_1c_2 & \cdots & r_1c_p \\ r_2c_1 & r_2c_2 & \cdots & r_2c_p \\ \vdots & \vdots & & \vdots \\ r_mc_1 & r_mc_2 & \cdots & r_mc_p \end{pmatrix}$$

Elimination by Block

One at a time $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$.

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

Block elimination

$$\left[\begin{array}{c|c} I & \mathbf{0} \\ \hline -CA^{-1} & I \end{array}\right]\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline \mathbf{0} & D - CA^{-1}B \end{array}\right].$$