Lecture 3

Linear Systems and Elimination

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Based on Dr. Ralph Chikhany's Slide

Strang Sections 2.1 – Vectors and Linear Equations and 2.2 – The Idea of Elimination

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text

Elimination

Systems of Equations

Example: Solve the system

$$x_1 + 2x_2 + 3x_3 = 6 \quad \leftarrow (1)$$
$$2x_1 + 5x_2 + 2x_3 = 4 \quad \leftarrow (2)$$
$$6x_1 - 3x_2 + x_3 = 2 \quad \leftarrow (3)$$

What strategies do you know?

3 × 3 sytem → (Simplify) 2 × 2 System → (Simplify) 1×1 system, $x_1 = \cdots$

Operations

1. (1) ⇔ c · (1) $cx_1 + 2cx_2 + 3cx_3 = 6c$ $c \neq 0$
2. Replace (2) with (1) + (2)

   (1) + (2): $3x_1 + 7x_2 + 5x_3 = 10$
3. Change the order of Equation

First Step

3x3 → 2x2

3x3	2x2	
x_1		
x_2	x_2	
x_3	x_3	2 Equation, no x_1

Using (1) to eliminate x_1 in (2), (3)

(2) − 2 × (1)

$0x_1 + x_2 - x_3 = -8$

(3) − 6 × (1)

$0x_1 - 15x_2 - 17x_3 = -30$

↑ 2x2 System

Value of x_1 ← Eq (1) ← Value of x_2, x_3 ← 2x2 System

General Case

① Using Eq (1) to Eliminate x_1 in (2) ⋯ (m) → ② Get a linear system of size $(m-1)\times(n-1)$

Solve it by another elimination Method

Using Eq (1) again, know x_1 ⟵ know solution $(x_2 \cdots x_n)$

Suppose we are given a system of m equations in n unknowns:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 && (1)\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 && (2)\\
&\vdots\\
a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n &= b_j && (j)\\
&\vdots\\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m && (m)
\end{aligned}
$$

$(2) - \frac{a_{21}}{a_{11}} \cdot (1)$

$(j) - \frac{a_{j1}}{a_{11}} (1)$

$a_{j2} - \frac{a_{j1}}{a_{11}} \cdot a_{12}$ $\quad$ $a_{jn} - \frac{a_{j1}}{a_{11}} \cdot a_{1n}$

$(m) - \frac{a_{m1}}{a_{11}} (1)$

our goal is to find $x_1, \ldots, x_n$.

The Process of Elimination

pivot

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \quad \text{first pivot}\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2\\
&\vdots\\
a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n &= b_j\\
&\vdots\\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

use pivot to eliminate the other Equation

To do that, we choose one equation which has a nonzero coefficient multiplying x_1 and use that to eliminate x_1 from all the remaining equations. This equation is referred to as the *first pivot*, and it could be any of the m equations in our system, e.g, the j^{th} equation $a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j$.

The Process of Elimination

What if my a_{11} is zero(?)

I just need $a_{11} \cdots a_{m1}$ one of them is not zero. change the order of Equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

this or any other equation with nonzero coefficient in front of ! x_1 can also be chosen as **first pivot**

$$\vdots$$

$$a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

To do that, we choose one equation which has a nonzero coefficient multiplying x_1 and use that to eliminate x_1 from all the remaining equations. This equation is referred to as the *first pivot*, and it could be any of the m equations in our system, e.g, the j^{th} equation $a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j$.

The Process of Elimination

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

this or any other equation with nonzero coefficient in front of ! % can also be chosen as **first pivot**

To do that, we choose one equation which has a nonzero coefficient multiplying x_1 and use that to eliminate x_1 from all the remaining equations. This equation is referred to as the *first pivot*, and it could be any of the m equations in our system, e.g, the j^{th} equation $a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j$.

The Process of Elimination

$$a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j$$

$$\bullet\, x_2 + \cdots + \bullet\, x_n = ⇤$$

$$\vdots$$

$$\bullet\, x_2 + \cdots + \bullet\, x_n = ⇧$$

the system after choosing the !'(equation as first pivot and using it to eliminate $% from the remaining equations

The Process of Elimination

$$a_{j\,1}\,x_1 + a_{j\,2}\,x_2 + \cdots + a_{j\,n}\,x_n = b_j$$

$$\bullet\, x_2 + \cdots + \bullet\, x_n = ⇤$$

pivot

$$\vdots$$

$$\bullet\, x_2 + \cdots + \bullet\, x_n = ⇧$$

choose **second pivot** with nonzero !) coefficient, and use it to eliminate !) from all remaining equations except the first pivot

Once we have eliminated x_1 from all equations except the first pivot, we move the pivot to the top, and leave it unaltered, then we choose another pivot from the remaining $m - 1$ equations, which has a nonzero coefficient multiplying x_2. We use this *second pivot* to eliminate x_2 from the $m - 2$ equations, i.e., all equations except the pivot equations (first and second).

The Process of Elimination

non zeros

$$
\begin{aligned}
a_{j1}x_1 + a_{j2}x_2 + a_{j3}x_3 + \cdots + a_{jn}x_n &= b_j \quad \text{first pivot}\\
\bullet\, x_2 + \bullet\, x_3 + \cdots + \bullet\, x_n &= \bullet \quad \text{second pivot}\\
\bullet\, x_3 + \cdots + \bullet\, x_n &= \bullet \quad \text{third pivot}\\
&\vdots\\
\bullet\, x_3 + \cdots + \bullet\, x_n &= \bullet
\end{aligned}
$$

zeros

$(m-2) \times (n-2)$

Once that is done, we move the second pivot and place it right under the first, and we leave it unaltered. We proceed by selecting a third pivot, which we use to eliminate x_3 from the remaining $m - 3$ equations.

Systems in Upper Triangular Form

Upper Triangular System/ Form

non-zero ↓

$$a_{j\,1}\,x_1 + a_{j\,2}\,x_2 + a_{j\,3}\,x_3 + \cdots + a_{j\,n}\,x_n = b_j$$

$$\bullet\, x_2 + \bullet\, x_3 + \cdots + \bullet\, x_n = \;?$$

$$\bullet\, x_3 + \cdots + \bullet\, x_n = \circ$$

$$\ddots$$

$$\bullet\, x_{n-1} + \bullet\, x_n = \;?$$

$$\bullet\, x_n = \;?$$

zeros!

Solve the Eq by reverse order

→ know x_1

$x_3 \cdots x_n$ → know x_3

x_n, x_{n-1} → known x_{n-1}

← only dependent on x_n → known x_n

We continue with this procedure, until the system is upper triangular. Once that is achieved, we can use the last equation to solve for x_n and then back-solve for all the remaining unknowns.

Example

Example: Solve the system

$$x_1 + 2x_2 + 3x_3 = 6 \quad (1)$$
$$2x_1 + 5x_2 + 2x_3 = 4 \quad (2)$$
$$6x_1 - 3x_2 + x_3 = 2 \quad (3)$$

Upper Triangular Form

$x_1 + 2x_2 + 3x_3 = 6$ (1)

$-x_2 + 4x_3 = 8$ (2')

$\cdots x_3 = \cdots$ (3'')

Solve 3x3

3x3 system

Using (1) to eliminate x_1

(2) − 2× (1)

$-x_2 + 4x_3 = 8$ (2')

(3) − 6× (1)

$15x_1 + 17x_3 = 34$ (3')

→ 2x2 system

Using (2') to eliminate x_2

Solve 2x2 system

(3') + 15× (2')

$(17 + 15\times 4)x_3 = 34 + 15\times 8$ (3'')

1x1 system

last Eq of Upper Triangular Form / 1x1 System

know x_3 from (3'')

1x1

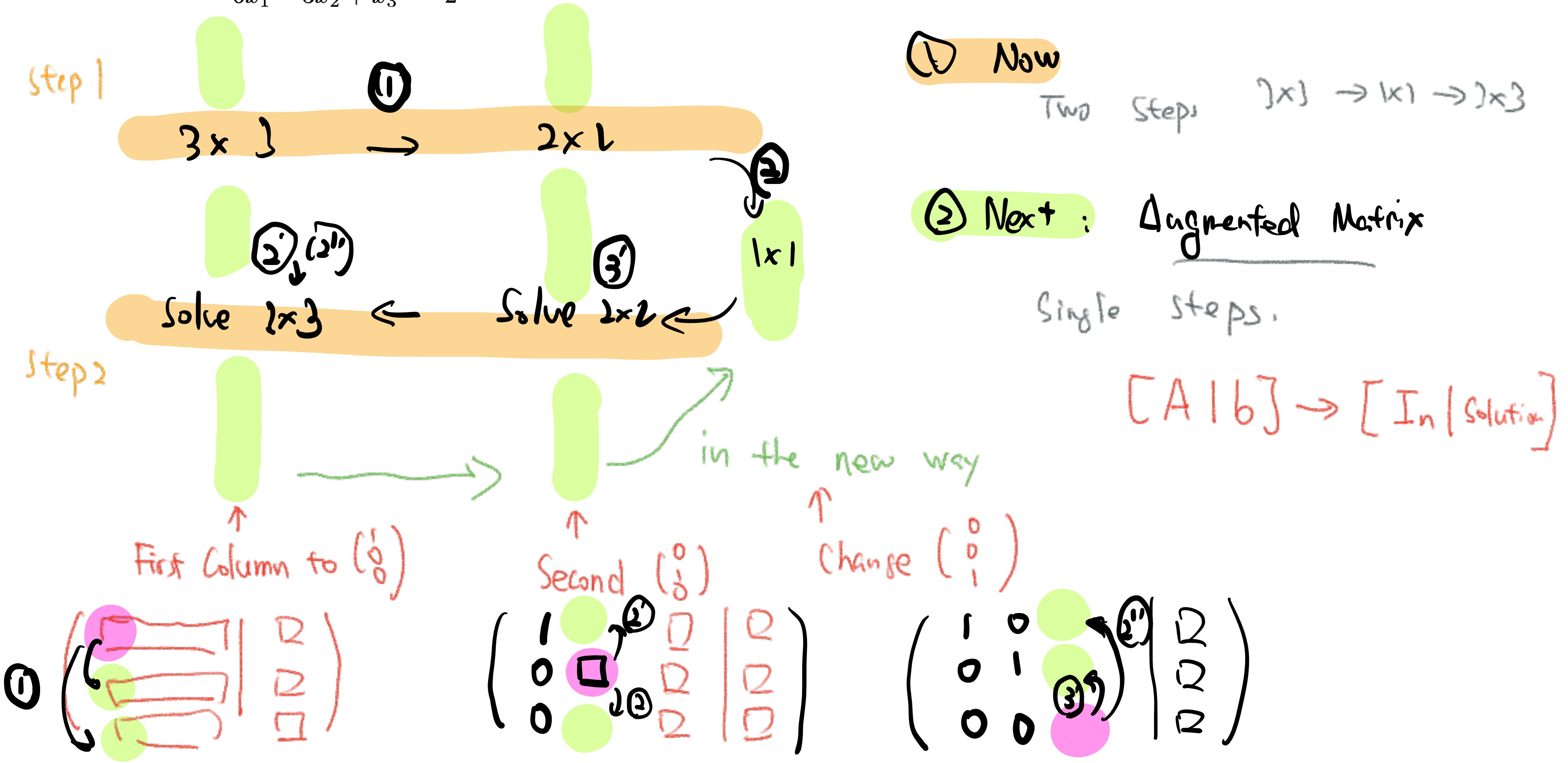
know x_2 from (2')

know x_1 from Equation (1)

Example

Example: Solve the system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 + 5x_2 + 2x_3 &= 4 \\ 6x_1 - 3x_2 + x_3 &= 2 \end{aligned}$$



This process is known as the Gauss-Jordan elimination method.
We can go even further to make the work more practical.

Gauss-Jordan

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

Elimination method: in what ways can you manipulate the equations?

- Multiply an equation by a nonzero number. **(scale)**
- Add a multiple of one equation to another. **(replacement)**
- Swap two equations. **(swap)**

Elimination

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

Elimination

It sure is a pain to have to write x, y, z, and $=$ over and over again.

Matrix notation: write just the numbers, in a box, instead!

$A\vec{x} = \vec{b}$. the vector $\vec{b}$ the same size of A's Column Vector

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned} \quad \xrightarrow{\text{becomes}} \quad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

$A \in \mathbb{R}^{m \times n}$ # m Equations # n unknowns

This is called an **(augmented) matrix**. Our equation manipulations become **elementary row operations**:

$[A \mid \vec{b}]$ Size of Augmented Matrix $m \times (n+1)$

- Multiply all entries in a row by a nonzero number. **(scale)**
- Add a multiple of each entry of one row to the corresponding entry in another. **(row replacement)**
- Swap two rows. **(swap)**

General Case

Aim: 3 operations to change $[A \mid \vec{b}]$ to $[I_n \mid \vec{c}]$

Then $\vec{c}$ is the solution of $A\vec{x} = \vec{b}$

Solution System

$$\begin{aligned} x_1 \quad\quad\quad &= c_1 \\ x_2 \quad\quad &= c_2 \\ \ddots \quad &\\ x_n &= c_n \end{aligned}$$

Suppose we are given a system of m equations in n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

This system can be written in matrix form as:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

in augmented form

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

← 3 operations to My Matrix
L.S Equivalent

Elimination. My Matrix → Upper Triangular

$$\left[\begin{array}{cccc|c} 1 & & 0 & & c_1 \\ & 1 & & & \vdots \\ & & \ddots & & \\ 0 & & & 1 & c_n \end{array}\right]$$

Identity Matrix I_n

Elimination

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

Start:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

Goal: we want our elimination method to eventually produce a system of equations like

$$\begin{aligned} x \qquad\qquad &= A \\ y \qquad &= B \\ z &= C \end{aligned} \qquad \text{or in matrix form,} \qquad \overset{I_3}{\left(\begin{array}{ccc|c} 1 & 0 & 0 & A \\ 0 & 1 & 0 & B \\ 0 & 0 & 1 & C \end{array}\right)}$$

So we need to do row operations that make the start matrix look like the end one.

Strategy: fiddle with it so we only have ones and zeros.

Elimination

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

Elimination

Example

Solve the system of equations

$$
\begin{aligned}
x + 2y + 3z &= 6\\
2x - 3y + 2z &= 14\\
3x + y - z &= -2
\end{aligned}
$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6\\ 2 & -3 & 2 & 14\\ 3 & 1 & -2 & -2 \end{array}\right) \begin{array}{l}(1)\\(2)\\(3)\end{array}
\;\rightarrow\;
\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6\\ 0 & 7 & 4 & -2\\ 0 & 5 & 10 & 20 \end{array}\right) \begin{array}{l}(1)\\2(1)-(2)\\3(1)-(3)\end{array}$$

Done!!

because we have 1 here 2 can't be the pivot

pivot, to eliminate 2, 5

can also use 5 as pivot

← switch my rows

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6\\ 0 & 5 \to 1 & 10 \to 2 & 20 \to 4\\ 0 & 7 & 4 & -2 \end{array}\right) \begin{array}{l}(1')\\(2')\\(3')\end{array}
\;\rightarrow\;
\left(\begin{array}{ccc|c} 1 & 0 & -1 & -2\\ 0 & 1 & 2 & 4\\ 0 & 0 & 1 & 3 \end{array}\right) \begin{array}{l}2(2')-(1')\\ \\7(2')-(3')\end{array}$$

Done

not change

both zero

The only thing can be used as pivot

Elimination

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

Elimination – Summary of the previous example

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right) \xrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 - 3R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right)$$

We want these to be zero.
So we subract multiples of the first row.

change first col to $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right) \xrightarrow{R_2 \longleftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 \div -5} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

$$\xrightarrow{R_1 = R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 + 7R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right)$$

We want these to be zero.

It would be nice if this were a 1.
We could divide by -7, but that would produce ugly fractions.

Let's swap the last two rows first.

change second col to $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Elimination

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right) \xrightarrow{R_3 = R_3 \div 10} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

We want these to be zero.

Let's make this a 1 first.

Third Column ⇓ $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\xrightarrow{R_1 = R_1 + R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 - 2R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Identity Matrix I_3 ← Solution

$$\xrightarrow{\text{translates into}} \begin{aligned} x &= 1 \\ y &= -2 \\ z &= 3 \end{aligned}$$

Success!

Check:

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned} \xrightarrow{\text{substitute solution}} \begin{aligned} 1 + 2\cdot(-2) + 3\cdot 3 &= 6 \\ 2\cdot 1 - 3\cdot(-2) + 2\cdot 3 &= 14 \\ 3\cdot 1 + (-2) - 3 &= -2 \end{aligned} \checkmark$$

Another Example

Example

Solve the system of equations

$$\begin{aligned} x + y &= 2 \\ 3x + 4y &= 5 \\ 4x + 5y &= 9 \end{aligned}$$

Another Example

Example

Solve the system of equations

$$\begin{aligned} x + y &= 2 \\ 3x + 4y &= 5 \\ 4x + 5y &= 9 \end{aligned}$$

Recall

Suppose we are given a system of m equations in n unknowns:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

This system can be written in matrix form as:

$$
\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

in augmented form →

$$
\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]
$$

Elimination Matrices

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \ddots & & & & & \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & & & \ddots & & & \\ 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & & & & & \ddots & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \ddots & & & & & \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & & & \ddots & & & \\ 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & & & & & \ddots & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

i-column ↓ j-column ↓

Change My Row j →

Change to c

linear Comb Two Rows

$[A \mid b] \rightarrow [E_{ji} A \mid E_{ji} b]$ → change My row (j) to (j) + c·(i)

Elimination Matrix

An elimination matrix is an $n \times n$ matrix which takes the $n \times n$ identity matrix and changes one of the zeros in the lower triangular or the upper triangular part of the identity matrix to some nonzero entry.

Matrix Multiply.

$E_{ji} \; A$

Matrix Matrix

Switch Two Rows   Row i ↔ Row j

$[P_{ij} A \mid P_{ij} b]$

Permutation Matrix

Elimination Matrices

Col i   Col j

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \ddots & & & & & \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & & & \ddots & & & \\ 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & & & & & \ddots & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

Row i

Row j

Elimination Matrices

Col i　　Col j

$$E_{ji} = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \ddots & & & & & \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & & & \ddots & & & \\ 0 & 0 & \dots & \star & \dots & 1 & \dots & 0 \\ \vdots & & & & & & \ddots & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

Row i

Row j ⇒ Replace (j) with (j) + * · (i)

Elimination Matrices

Col i Col j

$$E_{ji} = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \ddots & & & & & \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & & & \ddots & & & \\ 0 & 0 & \dots & \star & \dots & 1 & \dots & 0 \\ \vdots & & & & & & \ddots & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

Row i

Row j

$$E_{ji} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j + (\star \cdot x_i) \\ \vdots \\ x_n \end{bmatrix}$$

Elimination Matrices

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \star & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

When this matrix acts on a vector in $\mathbb{R}^n$, it adds $\star$ copies of the first row to the third row.

Elimination Matrices

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \star & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

a_{31}

When this matrix acts on a vector in $\mathbb{R}^n$, it adds $\star$ copies of the first row to the third row.

Change row (3) with (3) + $\star$ (1)

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \star & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \xrightarrow{E_{31}\,\vec{x}} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \star & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 + \star x_1 \\ \\ x_n \end{bmatrix}$$

Elimination Matrices

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \star & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

When this matrix acts on a vector in $\mathbb{R}^n$, it adds $\star$ copies of the first row to the third row.

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \star & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \xrightarrow{E_{31}\vec{x}} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \star & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 + (\star \cdot x_1) \\ \vdots \\ x_n \end{bmatrix}$$

Elimination Matrices

$$E_{23} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & * & & \vdots \\ 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & \dots & & 1 \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & \star & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \xrightarrow{E_{32}\,\vec{x}} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & \star & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 + (\star \cdot x_2) \\ \vdots \\ x_n \end{bmatrix}$$

$\uparrow a_{32}$

E_{32} is a lower Triangular Matrix ← $(i>j)$ E_{ij}

E_{23} is a upper Triangular Matrix ← $(i<j)$

Elimination Matrices

What does the matrix $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ do to the vector $\vec{x} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$ when it acts on it?

Elimination Matrices

What does the matrix $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ do to the vector $\vec{x} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}$ when it acts on it?

Note

Important

The process of doing row operations to a matrix does not change the solution set of the corresponding linear equations!

Definition

Two matrices are called **row equivalent** if one can be obtained from the other by doing some number of elementary row operations.

The linear equations of row-equivalent matrices have the *same solution set.*

In other words, the original equations

$$\begin{aligned} x + y &= 2 \\ 3x + 4y &= 5 \\ 4x + 5y &= 9 \end{aligned} \qquad \text{have the same solutions as} \qquad \begin{aligned} x + y &= 2 \\ y &= -1 \\ 0 &= 2 \end{aligned}$$

But the latter system obviously has no solutions (there is no way to make them all true), so our original system has no solutions either.

Definition

A system of equations is called **inconsistent** if it has no solution. It is **consistent** otherwise.