

Linear Algebra

Lecture 3 Linear Systems and Elimination

Yiping Lu Based on Dr. Ralph Chikhany's Slide



Strang Sections 2.1 – Vectors and Linear Equations and 2.2 – The Idea of Elimination

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text



Systems of Equations

Example: Solve the system $x_1 + 2x_2 + 3x_3 = 6$ $2x_1 + 5x_2 + 2x_3 = 4$ $6x_1 - 3x_2 + x_3 = 2$

What strategies do you know?

General Case

Suppose we are given a system of m equations in n unknowns:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{j1}x_1 + a_{j2}x_2 + \dots + a_{jn}x_n = b_j$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

our goal is to find x_1, \ldots, x_n .

 $\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}$

To do that, we choose one equation which has a nonzero coefficient multiplying x_1 and use that to eliminate x_1 from all the remaining equations. This equation is referred to as the *first pivot*, and it could be any of the *m* equations in our system, e.g, the j^{th} equation $a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j$.

 $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$ this or any other equation with nonzero coefficient in front of !% can also be chosen as **first pivot** $a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j$ $a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$

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$$a_{j1} x_1 + a_{j2} x_2 + \cdots + a_{jn} x_n = b_j$$

$$\cdot x_2 + \cdots + \cdot x_n = i \leftarrow$$

$$\vdots$$

$$\cdot x_2 + \cdots + \cdot x_n = \hat{\Box}$$

the system after choosing the !'(equation as first pivot and using it to eliminate \$% from the remaining equations

$$a_{j1} x_1 + a_{j2} x_2 + \cdots + a_{jn} x_n = b_j$$

$$\bullet x_2 + \cdots + \bullet x_n = \bullet$$

$$\vdots$$

$$\bullet x_2 + \cdots + \bullet x_n = \bullet$$

choose **second pivot** with nonzero!) coefficient, and use it to eliminate!) from all remaining equations except the first pivot

Once we have eliminated x_1 from all equations except the first pivot, we move the pivot to the top, and leave it unaltered, then we choose another pivot from the remaining m - 1 equations, which has a nonzero coefficient multiplying x_2 . We use this second pivot to eliminate x_2 from the m - 2 equations, i.e., all equations except the pivot equations (first and second).

$$a_{j1} x_1 + a_{j2} x_2 + a_{j3} x_3 + \cdots + a_{jn} x_n = b_j \quad \text{first pivot}$$

$$\cdot x_2 + \cdot x_3 + \cdots + \cdot x_n = i \leftarrow \quad \text{second pivot}$$

$$\cdot x_3 + \cdots + \cdot x_n = \circ \quad \text{third pivot}$$

$$\vdots$$

$$\cdot x_3 + \cdots + \cdot x_n = \boxtimes$$

Once that is done, we move the second pivot and place it right under the first, and we leave it unaltered. We proceed by selecting a third pivot, which we use to eliminate x_3 from the remaining m - 3 equations.

Systems in Upper Triangular Form

$$a_{j1} x_1 + a_{j2} x_2 + a_{j3} x_3 + \cdots + a_{jn} x_n = b_j$$

$$\cdot x_2 + \cdot x_3 + \cdots + \cdot x_n = i \leftarrow$$

$$\cdot x_3 + \cdots + \cdot x_n = \circ$$

$$\vdots$$

$$\cdot x_n = ?$$

We continue with this procedure, until the system is upper triangular. Once that is achieved, we can use the last equation to solve for x_n and then back-solve for all the remaining unknowns.

Example

Example: Solve the system $x_1 + 2x_2 + 3x_3 = 6$ $2x_1 + 5x_2 + 2x_3 = 4$

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This process is known as the Gauss-Jordan elimination method. We can go even further to make the work more practical.

Gauss-Jordan

Example

Solve the system of equations

$$x + 2y + 3z = 6
2x - 3y + 2z = 14
3x + y - z = -2$$

Elimination method: in what ways can you manipulate the equations?

- Multiply an equation by a nonzero number. (sca
 Add a multiple of one equation to another. (replacement
- ► Swap two equations.

(scale) (replacement) (swap)

Example

$$x + 2y + 3z = 6$$

$$2x - 3y + 2z = 14$$

$$3x + y - z = -2$$

It sure is a pain to have to write x, y, z, and = over and over again.

Matrix notation: write just the numbers, in a box, instead!

This is called an **(augmented) matrix**. Our equation manipulations become **elementary row operations**:

- ► Multiply all entries in a row by a nonzero number. (scale)
- Add a multiple of each entry of one row to the corresponding entry in another.
 (row replacement)
- ► Swap two rows. (swap)

General Case

Suppose we are given a system of m equations in n unknowns:

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$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

This system can be written in matrix form as:

a_{11}	a_{12}	 a_{1n}	x_1		$\begin{bmatrix} b_1 \end{bmatrix}$
a_{21}	a_{22}	 a_{2n}	x_2		b_2
:			:	=	
a_{m1}	a_{m2}	 a_{mn}	x_n		b_m

in augmented form	$egin{array}{c} a_{11} \ a_{21} \end{array}$	$a_{12} \ a_{22}$	 $a_{1n}\ a_{2n}$	$egin{array}{c c} b_1 \\ b_2 \end{array}$
	÷			:
	a_{m1}	a_{m2}	 a_{mn}	b_m

Example

Solve the system of equations

$$x + 2y + 3z = 6
2x - 3y + 2z = 14
3x + y - z = -2$$

Start:

$$\begin{pmatrix} 1 & 2 & 3 & | & 6 \\ 2 & -3 & 2 & | & 14 \\ 3 & 1 & -1 & | & -2 \end{pmatrix}$$

Goal: we want our elimination method to eventually produce a system of equations like

So we need to do row operations that make the start matrix look like the end one.

Strategy: fiddle with it so we only have ones and zeros.

Example

$$x + 2y + 3z = 6
2x - 3y + 2z = 14
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Example

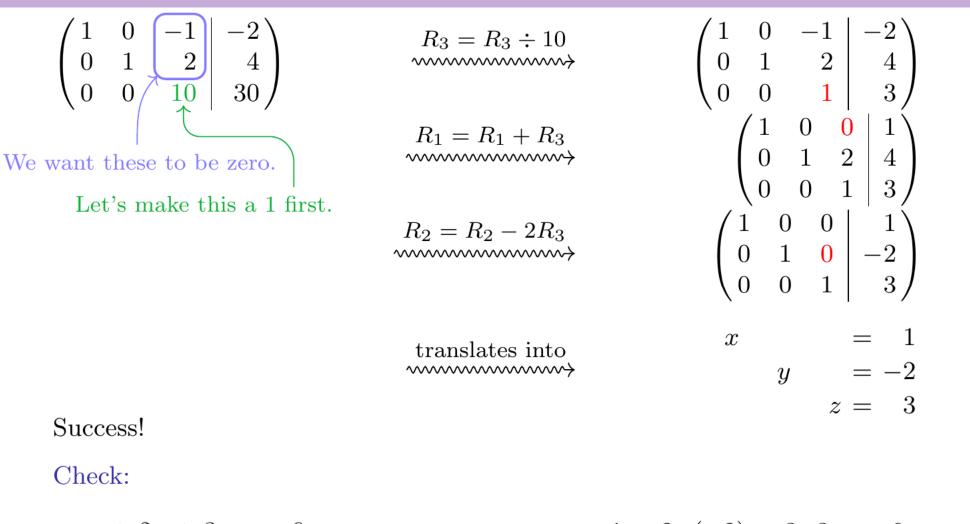
$$x + 2y + 3z = 6
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Elimination – Summary of the previous example

Let's swap the last two rows first. R_3

$$R_3 = R_3 + 7R_2$$

$$\begin{pmatrix} 1 & 2 & 3 & | & 6 \\ 0 & -7 & -4 & | & 2 \\ 3 & 1 & -1 & | & -2 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & 3 & | & 6 \\ 0 & -7 & -4 & | & 2 \\ 0 & -5 & -10 & | & -20 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & 3 & | & 6 \\ 0 & -5 & -10 & | & -20 \\ 0 & -7 & -4 & | & 2 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & 3 & | & 6 \\ 0 & 1 & 2 & | & 4 \\ 0 & -7 & -4 & | & 2 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & -1 & | & -2 \\ 0 & 1 & 2 & | & 4 \\ 0 & -7 & -4 & | & 2 \end{pmatrix}$$
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$$\begin{pmatrix} 1 & 0 & -1 & | & -2 \\ 0 & 1 & 2 & | & 4 \\ 0 & -7 & -4 & | & 2 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & -1 & | & -2 \\ 0 & 1 & 2 & | & 4 \\ 0 & 0 & 10 & | & 30 \end{pmatrix}$$



Another Example

Example

$$x + y = 2$$

$$3x + 4y = 5$$

$$4x + 5y = 9$$

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Recall

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$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

This system can be written in matrix form as:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$
in augmented form
$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

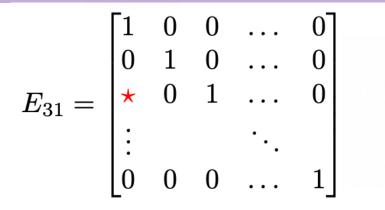


An elimination matrix is an $n \times n$ matrix which takes the $n \times n$ identity matrix and changes one of the zeros in the lower triangular or the upper triangular part of the identity matrix to some nonzero entry.

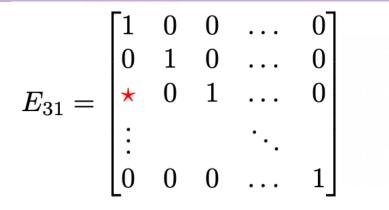
$$I = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \ddots & & & & & \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & & \ddots & & & \\ 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & & & \ddots & & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix} \operatorname{Rowij}$$

$$E_{ji} = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & & & & & \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & \dots & \star & \dots & 1 & \dots & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix} \operatorname{\mathsf{Rowij}}_{i}$$

$$E_{ji} = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \ddots & & & & & \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & & \ddots & & & \\ 0 & 0 & \dots & \star & \dots & 1 & \dots & 0 \\ \vdots & & & & \ddots & & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix} \mathbb{R} \circ \omega_{j} \qquad E_{ji} \begin{bmatrix} x_{1} \\ x_{2} \\ \vdots \\ x_{i} \\ \vdots \\ x_{j} \\ \vdots \\ x_{n} \end{bmatrix} = \begin{bmatrix} x_{1} \\ x_{2} \\ \vdots \\ x_{i} \\ \vdots \\ x_{j} \\ \vdots \\ x_{n} \end{bmatrix}$$

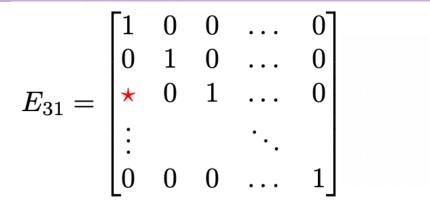


When this matrix acts on a vector in \mathbb{R}^n , it adds \star copies of the first row to the third row.

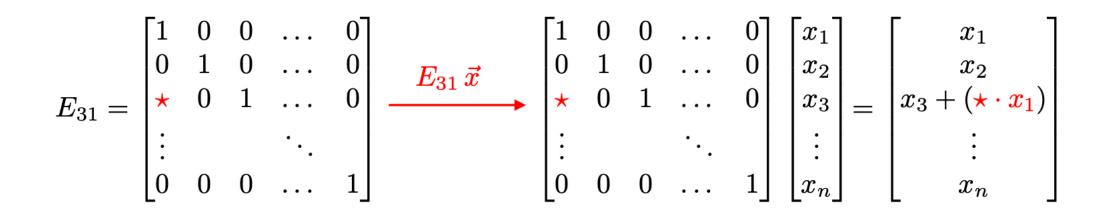


When this matrix acts on a vector in \mathbb{R}^n , it adds \star copies of the first row to the third row.

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \star & 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \xrightarrow{E_{31}\vec{x}} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \star & 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$



When this matrix acts on a vector in \mathbb{R}^n , it adds \star copies of the first row to the third row.



$$E_{32} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & \star & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \xrightarrow{E_{32} \vec{x}} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & \star & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

What does the matrix
$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
 do to the vector $\vec{x} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$ when it acts on it?

it acts on it?

What does the matrix
$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$
 do to the vector $\vec{x} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}$ when it acts on it?

Note

Important

The process of doing row operations to a matrix does not change the solution set of the corresponding linear equations!

Definition

Two matrices are called **row equivalent** if one can be obtained from the other by doing some number of elementary row operations.

The linear equations of row-equivalent matrices have the same solution set.

In other words, the original equations

x + y = 2		x + y = 2
3x + 4y = 5	have the same solutions as	y = -1
4x + 5y = 9		0 = 2

But the latter system obviously has no solutions (there is no way to make them all true), so our original system has no solutions either.

Definition

A system of equations is called **inconsistent** if it has no solution. It is **consistent** otherwise.