Lecture 3

Linear Systems and Elimination

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Based on Dr. Ralph Chikhany's Slide

Strang Sections 2.1 – Vectors and Linear Equations and 2.2 – The Idea of Elimination

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text

Elimination

Systems of Equations

Example: Solve the system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 + 5x_2 + 2x_3 &= 4 \\ 6x_1 - 3x_2 + x_3 &= 2 \end{aligned}$$

What strategies do you know?

General Case

Suppose we are given a system of m equations in n unknowns:

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1\\
&a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2\\
&\vdots\\
&a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j\\
&\vdots\\
&a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\end{aligned}
$$

our goal is to find $x_1, \ldots, x_n$.

The Process of Elimination

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$ first pivot

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

To do that, we choose one equation which has a nonzero coefficient multiplying x_1 and use that to eliminate x_1 from all the remaining equations. This equation is referred to as the *first pivot*, and it could be any of the m equations in our system, e.g, the j^{th} equation $a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j$.

The Process of Elimination

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$\boxed{a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2}$$

this or any other equation with nonzero coefficient in front of ! % can also be chosen as **first pivot**

$$\vdots$$

$$a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

To do that, we choose one equation which has a nonzero coefficient multiplying x_1 and use that to eliminate x_1 from all the remaining equations. This equation is referred to as the *first pivot*, and it could be any of the m equations in our system, e.g, the j^{th} equation $a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j$.

The Process of Elimination

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$\boxed{a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j}$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

this or any other equation with nonzero coefficient in front of ! % can also be chosen as **first pivot**

To do that, we choose one equation which has a nonzero coefficient multiplying x_1 and use that to eliminate x_1 from all the remaining equations. This equation is referred to as the *first pivot*, and it could be any of the m equations in our system, e.g, the j^{th} equation $a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j$.

The Process of Elimination

$$a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j$$

$$\bullet\, x_2 + \cdots + \bullet\, x_n = ⇤$$

$$\vdots$$

$$\bullet\, x_2 + \cdots + \bullet\, x_n = ⇧$$

the system after choosing the !'(equation as first pivot and using it to eliminate $% from the remaining equations

The Process of Elimination

$$a_{j\,1}\,x_1 + a_{j\,2}\,x_2 + \cdots + a_{j\;n}\,x_n = b_j$$

$$\bullet\; x_2 + \cdots + \bullet\; x_n = \mid\!\leftarrow$$

$$\vdots$$

$$\bullet\; x_2 + \cdots + \bullet\; x_n = \Uparrow$$

choose **second pivot** with nonzero !) coefficient, and use it to eliminate !) from all remaining equations except the first pivot

Once we have eliminated x_1 from all equations except the first pivot, we move the pivot to the top, and leave it unaltered, then we choose another pivot from the remaining $m - 1$ equations, which has a nonzero coefficient multiplying x_2. We use this *second pivot* to eliminate x_2 from the $m - 2$ equations, i.e., all equations except the pivot equations (first and second).

The Process of Elimination

$$a_{j1}x_1 + a_{j2}x_2 + a_{j3}x_3 + \cdots + a_{jn}x_n = b_j$$ **first pivot**

$$\bullet\, x_2 + \bullet\, x_3 + \cdots + \bullet\, x_n = \leftarrow$$ **second pivot**

$$\bullet\, x_3 + \cdots + \bullet\, x_n = \circ$$ **third pivot**

$$\vdots$$

$$\bullet\, x_3 + \cdots + \bullet\, x_n = \boxtimes$$

Once that is done, we move the second pivot and place it right under the first, and we leave it unaltered. We proceed by selecting a third pivot, which we use to eliminate x_3 from the remaining $m - 3$ equations.

Systems in Upper Triangular Form

$$
\begin{aligned}
a_{j1}x_1 + a_{j2}x_2 + a_{j3}x_3 + \cdots + a_{jn}x_n &= b_j \\
\bullet\, x_2 + \bullet\, x_3 + \cdots + \bullet\, x_n &= |\!\!\leftarrow \\
\bullet\, x_3 + \cdots + \bullet\, x_n &= \circ \\
&\ddots \\
\bullet\, x_n &= ?
\end{aligned}
$$

We continue with this procedure, until the system is upper triangular. Once that is achieved, we can use the last equation to solve for x_n and then back-solve for all the remaining unknowns.

Example

Example: Solve the system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 + 5x_2 + 2x_3 &= 4 \\ 6x_1 - 3x_2 + x_3 &= 2 \end{aligned}$$

Example

Example: Solve the system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 + 5x_2 + 2x_3 &= 4 \\ 6x_1 - 3x_2 + x_3 &= 2 \end{aligned}$$

This process is known as the Gauss-Jordan elimination method.
We can go even further to make the work more practical.

Gauss-Jordan

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

Elimination method: in what ways can you manipulate the equations?

- Multiply an equation by a nonzero number. **(scale)**
- Add a multiple of one equation to another. **(replacement)**
- Swap two equations. **(swap)**

Elimination

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

Elimination

It sure is a pain to have to write x, y, z, and $=$ over and over again.

Matrix notation: write just the numbers, in a box, instead!

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned} \quad \overset{\text{becomes}}{\rightsquigarrow} \quad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

This is called an **(augmented) matrix**. Our equation manipulations become **elementary row operations**:

- Multiply all entries in a row by a nonzero number. **(scale)**
- Add a multiple of each entry of one row to the corresponding entry in another. **(row replacement)**
- Swap two rows. **(swap)**

General Case

Suppose we are given a system of m equations in n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

This system can be written in matrix form as:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

in augmented form →

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

Elimination

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

Start:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

Goal: we want our elimination method to eventually produce a system of equations like

$$\begin{aligned} x \qquad\qquad &= A \\ y \qquad &= B \\ z &= C \end{aligned} \qquad \text{or in matrix form,} \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & A \\ 0 & 1 & 0 & B \\ 0 & 0 & 1 & C \end{array}\right)$$

So we need to do row operations that make the start matrix look like the end one.

Strategy: fiddle with it so we only have ones and zeros.

Elimination

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

Elimination

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Elimination

Example

Solve the system of equations

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Elimination – Summary of the previous example

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right) \xrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 - 3R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right)$$

We want these to be zero.
So we subract multiples of the first row.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right) \xrightarrow{R_2 \longleftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 \div -5} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

$$\xrightarrow{R_1 = R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 + 7R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right)$$

We want these to be zero.

It would be nice if this were a 1.
We could divide by -7, but that would produce ugly fractions.

Let's swap the last two rows first.

Elimination

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right) \xrightarrow{R_3 = R_3 \div 10} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

We want these to be zero.

Let's make this a 1 first.

$$\xrightarrow{R_1 = R_1 + R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 - 2R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{\text{translates into}} \begin{aligned} x \quad\quad &= 1 \\ y \quad &= -2 \\ z &= 3 \end{aligned}$$

Success!

Check:

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned} \xrightarrow{\text{substitute solution}} \begin{aligned} 1 + 2\cdot(-2) + 3\cdot 3 &= 6 \\ 2\cdot 1 - 3\cdot(-2) + 2\cdot 3 &= 14 \\ 3\cdot 1 + (-2) - 3 &= -2 \end{aligned}$$

✔

Another Example

Example

Solve the system of equations

$$\begin{aligned} x + y &= 2 \\ 3x + 4y &= 5 \\ 4x + 5y &= 9 \end{aligned}$$

Another Example

Example

Solve the system of equations

$$\begin{aligned} x + y &= 2 \\ 3x + 4y &= 5 \\ 4x + 5y &= 9 \end{aligned}$$

Recall

Suppose we are given a system of m equations in n unknowns:

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This system can be written in matrix form as:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

in augmented form →

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

Elimination Matrices

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \ddots & & & & & \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & & & \ddots & & & \\ 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & & & & & \ddots & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \ddots & & & & & \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & & & \ddots & & & \\ 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & & & & & \ddots & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

An elimination matrix is an $n \times n$ matrix which takes the $n \times n$ identity matrix and changes one of the zeros in the lower triangular or the upper triangular part of the identity matrix to some nonzero entry.

Elimination Matrices

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \ddots & & & & & \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & & & \ddots & & & \\ 0 & 0 & \dots & \textcircled{0} & \dots & 1 & \dots & 0 \\ \vdots & & & & & & \ddots & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

Col i, Col j (columns); Row i, Row j (rows)

Elimination Matrices

Col i Col j

$$E_{ji} = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \ddots & & & & & \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & & & \ddots & & & \\ 0 & 0 & \dots & \star & \dots & 1 & \dots & 0 \\ \vdots & & & & & & \ddots & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ \text{Row } i \\ \\ \text{Row } j \\ \\ \\ \end{matrix}$$

Elimination Matrices

Col i Col j

$$E_{ji} = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \ddots & & & & & \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & & & \ddots & & & \\ 0 & 0 & \dots & \star & \dots & 1 & \dots & 0 \\ \vdots & & & & & & \ddots & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ \text{Row } i \\ \\ \text{Row } j \\ \\ \\ \end{matrix}$$

$$E_{ji} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j + (\star \cdot x_i) \\ \vdots \\ x_n \end{bmatrix}$$

Elimination Matrices

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \star & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

When this matrix acts on a vector in $\mathbb{R}^n$, it adds $\star$ copies of the first row to the third row.

Elimination Matrices

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \star & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

When this matrix acts on a vector in $\mathbb{R}^n$, it adds $\star$ copies of the first row to the third row.

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \star & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \xrightarrow{E_{31}\,\vec{x}} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \star & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

Elimination Matrices

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \star & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

When this matrix acts on a vector in $\mathbb{R}^n$, it adds $\star$ copies of the first row to the third row.

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \star & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \xrightarrow{E_{31}\,\vec{x}} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \star & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 + (\star \cdot x_1) \\ \vdots \\ x_n \end{bmatrix}$$

Elimination Matrices

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & \star & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \xrightarrow{E_{32}\,\vec{x}} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & \star & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 + (\star \cdot x_2) \\ \vdots \\ x_n \end{bmatrix}$$

Elimination Matrices

What does the matrix $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ do to the vector $\vec{x} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$ when it acts on it?

Elimination Matrices

What does the matrix $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ do to the vector $\vec{x} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}$ when it acts on it?

Note

Important

The process of doing row operations to a matrix does not change the solution set of the corresponding linear equations!

Definition

Two matrices are called **row equivalent** if one can be obtained from the other by doing some number of elementary row operations.

The linear equations of row-equivalent matrices have the *same solution set.*

In other words, the original equations

$$\begin{aligned} x + y &= 2 \\ 3x + 4y &= 5 \\ 4x + 5y &= 9 \end{aligned} \qquad \text{have the same solutions as} \qquad \begin{aligned} x + y &= 2 \\ y &= -1 \\ 0 &= 2 \end{aligned}$$

But the latter system obviously has no solutions (there is no way to make them all true), so our original system has no solutions either.

Definition

A system of equations is called **inconsistent** if it has no solution. It is **consistent** otherwise.