Lecture 4
Matrix Operations

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Based on Dr. Ralph Chikhany's Slide

Reminders

- Get access to Gradescope, Campuswire.
- Obtain the textbook.
- Problem Set 1 due by 11.59 pm on Friday (NY time). This Friday
  - ✓ Late work policy applies.
- Recap Quiz 2 due by 11.59 pm on Sunday (NY time).
  - ❖ Late work policy does not apply.
- Recap Quiz is timed.
  - ❑ Once you start, you have ~~60~~ 45 minutes to finish it (even if you close the tab)

Recap

In general, n vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly dependent if

$$\rightarrow \quad c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$$

for scalars $c_1, c_2, \ldots, c_n$ not all zero. at least one of $c_1 \cdots c_n$ is not zero

Example: $v_1 = (1, -1, 0)$, $v_2 = (-2, 2, 0)$, $v_3 = (0, 0, 1)$

$$\underbrace{2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}}_{\text{non-zero}} + 0 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

↳ linear dependent

have a zero

Recap

Linear system $Ax = b$ $Ax = \underbrace{x_1 \cdot v_1 + x_2 v_2 \cdots x_n v_n}_{\text{L.C. of Column Vectors}}$

- have solution if and only if $b \in \text{span}\{c_1, \cdots, c_n\}$ where $c_1, \cdots, c_n$ are column vectors of matrix A

- have a unique solution if and only if matrix A have an inverse matrix A^{-1}. The unique solution is $\boxed{x = A^{-1}b}$. In this case, A must be a square matrix and $c_1, \cdots, c_n$ are linear independent.

If A have an inverse
1. A is square matrix $\mathbb{R}^{n \times n}$
2. All n column vector ($\mathbb{R}^n$) are linear independent.

1) A is "fat" Matrix # Col n (→ # unknown) > # row m (→ # Eq) $A \in \mathbb{R}^{m \times n}$

$[\,|\ |\ \cdots\ |\,]$ $\underbrace{n \text{ column vectors}}_{\mathbb{R}^m}$ m row vectors. ⇒ Column vectors must be linear dependent

2) A is "tall" matrix # col n (→ # unknown) < # row m (→ # Eq) there exist $\vec{b} \in \mathbb{R}^m$ s.t. $Ax \neq b$

$[\,|\ |\ \cdots\ |\,]$ We don't have enough Column vector $\mathbb{R}^m$ ⇒ Can't span the whole $\mathbb{R}^m$

n # column < n

Recap

Linear system $Ax = b$

- have solution if and only if $b \in \text{span}\{c_1, \cdots, c_n\}$ where $c_1, \cdots, c_n$ are column vectors of matrix A
- have a unique solution if and only if matrix A have an inverse matrix A^{-1}. The unique solution is $x = A^{-1}b$. In this case, A must be a square matrix and $c_1, \cdots, c_n$ are linear independent.

$\{\vec{v}_1 \cdots \vec{v}_n\}$ $\quad \vec{v}_i \in \mathbb{R}^m$

① $n > m$ linear dependent

② $n < m$

linear independent → biggest span

linear dependent

$\Leftrightarrow \vec{v}_1 \in \text{span}\{\vec{v}_2 \cdots \vec{v}_n\}$

$\vec{v}_1 = c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$

$n=1$ 1 vector line in $\mathbb{R}^m$ $\quad n=3$ 3 vector space in $\mathbb{R}^m$

$n=2$ 2 vector plane in $\mathbb{R}^m$ ...

Strang Sections 2.3 – Elimination Using Matrices and 2.4 – Rules for Matrix Operations

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text

Permutation Matrices

Recall

$$I = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \xrightarrow{I\,\vec{x}} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

identity Matrix

$I_n \vec{x} = \vec{x}$

Permutation Matrices

Inverse of P_{ij} is P_{ij}

$$
P_{ij} = \begin{bmatrix}
1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\
0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\
\vdots & & & & & & & \\
0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\
\vdots & & & & & & & \\
0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\
\vdots & & & & & & & \\
0 & 0 & \dots & 0 & \dots & 0 & \dots & 1
\end{bmatrix}
\begin{array}{l}
\\ \\ \\ \leftarrow i\text{-th row} \\ \\ \leftarrow j\text{-th row} \\ \\ \\
\end{array}
$$

$$
\begin{array}{cc} \uparrow & \uparrow \\ i\text{-column} & j\text{-column} \end{array}
$$

$$
P_{ij} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix}
$$

Example $\mathbb{R}^5$ switch 2-th element / 4-th element of the vector

$I_5 \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$$
P_{24} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_4 \\ x_3 \\ x_2 \\ x_5 \end{bmatrix}
$$

Permutation Matrices

$$P_{ij} = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

$$P_{ij} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix}$$

Switch 1-th row of A
3-rd row

$Ax = b \quad \rightarrow \quad P_{31} A x = P_{31} b$

$$P_{31} = \begin{bmatrix} 0 & 0 & 1 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = b_1$

$\vdots \qquad \vdots$

$a_{31} x_1 + a_{32} x_2 + \dots + a_{3n} x_n = b_3$

$\vdots$

$a_{m1} x_1 + \dots + a_{mn} x_n = b_m$

Permutation Matrices

$$P_{ij} = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & & & & & & \\ \color{blue}0 & \color{blue}0 & \color{blue}\dots & \color{blue}0 & \color{blue}\dots & \color{blue}1 & \color{blue}\dots & \color{blue}0 \\ \vdots & & & & & & & \\ \color{red}0 & \color{red}0 & \color{red}\dots & \color{red}1 & \color{red}\dots & \color{red}0 & \color{red}\dots & \color{red}0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

$$P_{ij} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \color{blue}x_j \\ \vdots \\ \color{red}x_i \\ \vdots \\ x_n \end{bmatrix}$$

$$P_{31} = \begin{bmatrix} \color{blue}0 & \color{blue}0 & \color{blue}1 & \color{blue}\dots & \color{blue}0 \\ 0 & 1 & 0 & \dots & 0 \\ \color{red}1 & \color{red}0 & \color{red}0 & \color{red}\dots & \color{red}0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \xrightarrow{\color{red}P_{31}\,\vec{x}} \begin{bmatrix} \color{blue}0 & \color{blue}0 & \color{blue}1 & \color{blue}\dots & \color{blue}0 \\ 0 & 1 & 0 & \dots & 0 \\ \color{red}1 & \color{red}0 & \color{red}0 & \color{red}\dots & \color{red}0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

Matrix Operations

Recall

Let A be an $m \times n$ matrix.

We write a_{ij} for the entry in the ith row and the jth column. It is called the **ijth entry** of the matrix.

$$\begin{array}{cc} \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix} & i\text{th row} \\ j\text{th column} & \end{array}$$

The entries $a_{11}, a_{22}, a_{33}, \ldots$ are the **diagonal entries**; they form the **main diagonal** of the matrix.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

A **diagonal matrix** is a *square* matrix whose only nonzero entries are on the main diagonal.

$$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

The $n \times n$ **identity matrix** I_n is the diagonal matrix with all diagonal entries equal to 1. It is special because $I_n v = v$ for *all* v in $\mathbf{R}^n$.

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Recall

The **zero matrix** (of size $m \times n$) is the $m \times n$ matrix 0 with all zero entries.

$$0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix A^T whose rows are the columns of A. In other words, the ij entry of A^T is a_{ji}.

$$\overset{A}{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}} \rightsquigarrow \overset{A^T}{\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}}$$

flip

$A: \mathbb{R}^{m\times n} \rightsquigarrow A^T: \mathbb{R}^{n\times m}$

Matrix Addition and Scalar Multiplication

You add two matrices component by component, like with vectors.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{pmatrix}$$

Note you can only add two matrices *of the same size.*

A+ B : A and B have the same size

You multiply a matrix by a scalar by multiplying each component, like with vectors.

$$c \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} & ca_{13} \\ ca_{21} & ca_{22} & ca_{23} \end{pmatrix}.$$

These satisfy the expected rules, like with vectors:

$$\begin{aligned} A + B &= B + A & (A+B)+C &= A+(B+C) \\ c(A+B) &= cA + cB & (c+d)A &= cA + dA \\ (cd)A &= c(dA) & A + 0 &= A \end{aligned}$$

Matrix Multiplication

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & & \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & \dots & b_{1m} \\ b_{21} & \dots & b_{2m} \\ \vdots & & \\ b_{l1} & \dots & b_{lm} \end{bmatrix} \quad C = \begin{bmatrix} c_{11} & \dots & c_{1k} \\ c_{21} & \dots & c_{2k} \\ \vdots & & \\ c_{n1} & \dots & c_{nk} \end{bmatrix}$$

$\mathbb{R}^{m\times k}$ $\mathbb{R}^{k\times m}$

Rlue. $A \in \mathbb{R}^{m\times n}$ can oly muliply with matrix look like $\mathbb{R}^{n\times k}$

Example 1) $A \in \mathbb{R}^{m\times n}$ vector $\mathbb{R}^n$ n row - 1 column. $\rightarrow \mathbb{R}^{n\times 1}$ $\mathbb{R}^{m\times n} \cdot \mathbb{R}^{n\times 1}$

2) $m \neq n \neq k$ $A \in \mathbb{R}^{m\times n}$ $B \in \mathbb{R}^{l\times m}$ $C \in \mathbb{R}^{n\times k}$. I can do AC. $C^T A^T$

① $A \cdot B$ $\mathbb{R}^{m\times n}$ $\mathbb{R}^{l\times m}$ × $B \cdot A$ $\mathbb{R}^{l\times m}$ $\mathbb{R}^{m\times n}$ $\rightarrow \mathbb{R}^{l\times n}$ ✓ can't do CA $A^T C^T$

$A^T \cdot B^T$ $\mathbb{R}^{n\times m}$ $B^{m\times l}$ ✓ $\hookrightarrow \mathbb{R}^{n\times l}$ $B^T \cdot A^T$ $\mathbb{R}^{m\times l}$ $\mathbb{R}^{n\times m}$ ×

Fact. we can do $B \cdot A$ doesn't mean we can do $A \cdot B$

but is means we can do $A^T \cdot B^T$

Matrix Multiplication

Beware: matrix multiplication is more subtle than addition and scalar multiplication.

must be equal

Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix with columns $v_1, v_2, \ldots, v_p$:

$v_i \in \mathbb{R}^n$ $\quad$ $v_i \in \mathbb{R}^n$

$$B = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{pmatrix}.$$

The **product** AB is the $m \times p$ matrix with columns $Av_1, Av_2, \ldots, Av_p$:

The equality is a definition

$$AB \stackrel{\text{def}}{=} \begin{pmatrix} | & | & & | \\ Av_1 & Av_2 & \cdots & Av_p \\ | & | & & | \end{pmatrix}.$$

$\mathbb{R}^m$ $\quad$ $\mathbb{R}^m$ $\quad$ $\mathbb{R}^m$

$\mathbb{R}^{m\times n}$ $\quad$ $\mathbb{R}^n$

$A\,B \in \mathbb{R}^{m\times p}$

In order for $Av_1, Av_2, \ldots, Av_p$ to make sense, the number of columns of A has to be the same as the number of rows of B.

$\mathbb{R}^{m\times n} \cdot \mathbb{R}^{n\times p} \to \mathbb{R}^{m\times p}$

Example

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}}_{2\times 3} \overbrace{\begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix}}^{3\times 2} = \left(\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix} \right) \to 2\times 2$$

$$= \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}$$

Example. $P_{13} A$

switch the first row

the third row

$$P_{13}\begin{pmatrix} \vec{v_1} & \cdots & \vec{v_n} \end{pmatrix} = \begin{pmatrix} P_{13}\vec{v_1} & \cdots & P_{13}\vec{v_n} \end{pmatrix}$$

Matrix Multiplication

A row vector of length n times a column vector of length n is a scalar:

$$\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + \cdots + a_n b_n.$$

Matrix Multiplication

A row vector of length n times a column vector of length n is a scalar:

$$\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + \cdots + a_n b_n.$$

Another way of multiplying a matrix by a vector is:

$$Ax = \begin{pmatrix} \text{—}\, r_1 \,\text{—} \\ \vdots \\ \text{—}\, r_m \,\text{—} \end{pmatrix} x = \begin{pmatrix} r_1 x \\ \vdots \\ r_m x \end{pmatrix}.$$

Matrix Multiplication

A row vector of length n times a column vector of length n is a scalar:

$$\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + \cdots + a_n b_n.$$

Another way of multiplying a matrix by a vector is:

$$Ax = \begin{pmatrix} — r_1 — \\ \vdots \\ — r_m — \end{pmatrix} x = \begin{pmatrix} r_1 x \\ \vdots \\ r_m x \end{pmatrix}.$$

On the other hand, you multiply two matrices by

$$AB = A \begin{pmatrix} | & & | \\ c_1 & \cdots & c_p \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ Ac_1 & \cdots & Ac_p \\ | & & | \end{pmatrix}.$$

Matrix Multiplication

A row vector of length n times a column vector of length n is a scalar:

$$\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + \cdots + a_n b_n .$$

Another way of multiplying a matrix by a vector is:

$$Ax = \begin{pmatrix} — r_1 — \\ \vdots \\ — r_m — \end{pmatrix} x = \begin{pmatrix} r_1 x \\ \vdots \\ r_m x \end{pmatrix} .$$

On the other hand, you multiply two matrices by

$$AB = A \begin{pmatrix} | & & | \\ c_1 & \cdots & c_p \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ Ac_1 & \cdots & Ac_p \\ | & & | \end{pmatrix} .$$

It follows that

$$\boxed{AB = \begin{pmatrix} — r_1 — \\ \vdots \\ — r_m — \end{pmatrix} \begin{pmatrix} | & & | \\ c_1 & \cdots & c_p \\ | & & | \end{pmatrix} = \begin{pmatrix} r_1 c_1 & r_2 c_2 & \cdots & r_1 c_p \\ r_2 c_1 & r_2 c_2 & \cdots & r_2 c_p \\ \vdots & \vdots & & \vdots \\ r_m c_1 & r_m c_2 & \cdots & r_m c_p \end{pmatrix}}$$

$\mathbb{R}^n$ (r_1), $\mathbb{R}^{m \times n}$, $\mathbb{R}^{n \times p}$, $\mathbb{R}^n$ (c_1), $\mathbb{R}^n$ (c_p)

i-th row. j-th col. is

$$\underbrace{\vec{r_i}^T}_{\mathbb{R}^n} \underbrace{c_j}_{\mathbb{R}^p}$$

m-rows

P-cols.

Matrix Multiplication

The ij entry of $C = AB$ is the ith row of A times the jth column of B:

$$c_{ij} = (AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

This is how everybody on the planet actually computes AB. Diagram ($AB = C$):

$$\begin{pmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ik} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mk} & \cdots & a_{mn} \end{pmatrix} \begin{array}{c} \\ \\ i\text{th row} \\ \\ \\ \end{array} \cdot \begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{k1} & \cdots & b_{kj} & \cdots & b_{kp} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mp} \end{pmatrix}$$

jth column

ij entry

Matrix Multiplication

The ij entry of $C = AB$ is the ith row of A times the jth column of B:

$$c_{ij} = (AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

This is how everybody on the planet actually computes AB. Diagram ($AB = C$):

$$\begin{array}{c} \begin{pmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ik} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mk} & \cdots & a_{mn} \end{pmatrix} \text{ith row} \end{array} \cdot \begin{array}{c} \begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{k1} & \cdots & b_{kj} & \cdots & b_{kp} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{pmatrix} \\ j\text{th column} \end{array} = \begin{array}{c} \begin{pmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mp} \end{pmatrix} \\ ij\text{ entry} \end{array}$$

Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 2\cdot 2 + 3\cdot 3 & \square \\ \square & \square \end{pmatrix} = \begin{pmatrix} 14 & \square \\ \square & \square \end{pmatrix}$$

(1.2.3) (-3,-2,-1) = -10

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} \square & \square \\ 4\cdot 1 + 5\cdot 2 + 6\cdot 3 & \square \end{pmatrix} = \begin{pmatrix} \square & \square \\ 32 & \square \end{pmatrix}$$

(4 5 6) · (-3 -2 -1)

= -28

Matrix-Matrix and Matrix-Vector

Matrix vector multiplication is a Matrix Matrix multiplication

$A \in \mathbb{R}^{m\times n}$ $\quad \vec{v} \in \mathbb{R}^{n}$ ($\mathbb{R}^{n\times 1}$) $\Rightarrow \underline{\mathbb{R}^{m\times 1}} \rightarrow \mathbb{R}^{m}$- vector

$r_i^T B$

$$\begin{bmatrix} \vec{r}_1^{\,T} \\ \vdots \\ \vec{r}_m^{\,T} \end{bmatrix} [\vec{v}_1 \cdots \vec{v}_k] = \begin{bmatrix} \vec{r}_1^{\,T}\vec{v}_1 & \vec{r}_1^{\,T}\vec{v}_2 & \cdots & \vec{r}_1^{\,T}\vec{v}_k \\ \vdots & \vdots & & \\ \vec{r}_m^{\,T}\vec{v}_1 & \vec{r}_m^{\,T}\vec{v}_2 & & \vec{r}_m^{\,T}\vec{v}_k \end{bmatrix}$$

$$\begin{bmatrix} \vec{r}_1^{\,T} \\ \vec{r}_2^{\,T} \\ \cdots \\ \vec{r}_k^{\,T} \end{bmatrix} B = \begin{bmatrix} \vec{r}_1^{\,T} B \\ \vec{r}_2^{\,T} B \\ \cdots \\ \vec{r}_k^{\,T} B \end{bmatrix}$$

$r_m^T B$

Check after Class.

check

$$A[\vec{v}_1, \cdots, \vec{v}_k] = [A\vec{v}_1, \cdots, A\vec{v}_k]$$

$$\begin{bmatrix} \vec{r}_1^{\,T} \\ \vdots \\ \vec{r}_m^{\,T} \end{bmatrix} [\vec{v}_1 \cdots \vec{v}_k] = \begin{bmatrix} \vec{r}_1^{\,T}\vec{v}_1 & \vec{r}_1^{\,T}\vec{v}_2 & \cdots & \vec{r}_1^{\,T}\vec{v}_k \\ \vdots & \vdots & & \\ \vec{r}_m^{\,T}\vec{v}_1 & \vec{r}_m^{\,T}\vec{v}_2 & & \vec{r}_m^{\,T}\vec{v}_k \end{bmatrix}$$

$A\vec{v}_1 \qquad A\vec{v}_2 \qquad A\vec{v}_k$

Recap in the Next Class

Matrix Multiplication

Let $A = \begin{bmatrix} 1 & 2 & 5 & 3 \\ 3 & -1 & 0 & -3 \\ 2 & -2 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 0 \\ 4 & 1 & 3 \\ 0 & -2 & 1 \end{bmatrix}$.

Compute AB and BA (if possible).

$A \in \mathbb{R}^{3\times4}$ $\quad$ $B \in \mathbb{R}^{4\times3}$

$AB \quad \checkmark \quad \mathbb{R}^{3\times3}$

$BA \cdot \quad \checkmark \quad \mathbb{R}^{4\times4} \quad (\mathbb{R}^{4\times3} \quad \mathbb{R}^{3\times4})$

$$\begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 0 \\ 4 & 1 & 3 \\ 0 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 5 & 3 \\ 3 & -1 & 0 & -3 \\ 2 & -2 & 1 & 0 \end{bmatrix} = \begin{pmatrix} \boxed{4} & \boxed{2} & \boxed{11} & \boxed{6} \\ \boxed{2} & \boxed{-3} & \boxed{-5} & \boxed{-6} \\ \boxed{13} & \boxed{1} & \boxed{23} & \boxed{9} \\ \boxed{-4} & \boxed{0} & \boxed{1} & \boxed{6} \end{pmatrix}$$

Matrix Multiplication

$\mathbb{R}^{k\times n}\ \mathbb{R}^{n\times p} \to \mathbb{R}^{k\times p}$

$\mathbb{R}^{n\times m}\ \mathbb{R}^{m\times p}\qquad \mathbb{R}^{k\times m}\ \mathbb{R}^{m\times p}$

Mostly matrix multiplication works like you'd expect. Suppose A has size $m \times n$, and that the other matrices below have the right size to make multiplication work.

$\mathbb{R}^{k\times n}\ \mathbb{R}^{n\times m}$

← easy to check by the column view

$$A(BC) = (AB)C \qquad A(B+C) = (AB + AC)$$

check: row reivew → $(B + C)A = BA + CA$ $\qquad c(AB) = (cA)B$

$$c(AB) = A(cB) \qquad I_n A = A$$

$$AI_m = A$$

Most of these are easy to verify.

Matrix Multiplication

Warnings!

- AB is usually not equal to BA.

$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$ $BA = \begin{pmatrix} 5 & 5 \\ 7 & 7 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$ $AB = \begin{pmatrix} 5 & 7 \\ 5 & 7 \end{pmatrix}$

In fact, AB may be defined when BA is not.

- $AB = AC$ does not imply $B = C$, even if $A \neq 0$.

$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$, $C = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$

$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $AC = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

A's row is linear dependent

- $AB = 0$ does not imply $A = 0$ or $B = 0$. (0)

$AB = 0 \not\Rightarrow A = 0$ or $B = 0$

$AB = AC \Leftrightarrow AB - AC = 0 \Leftrightarrow A(B-C) = 0 \not\Rightarrow A = 0$ or $B = C$

$\Uparrow$

If A have an inverse.

$$AB = AC \Rightarrow A^{-1}AB = A^{-1}AC$$

$$\Rightarrow B = C$$

Next Time Recap

$$A^{-1}A = AA^{-1} = I_n$$

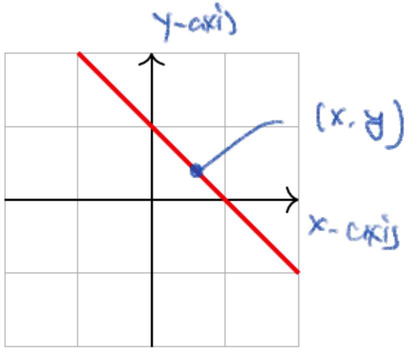
Systems of Equations

Systems of Equations

What does the solution set of a linear equation look like?

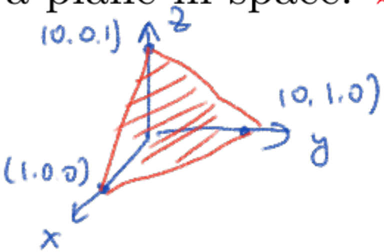
- $x + y = 1$

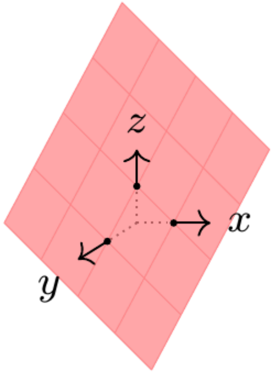
  $\rightsquigarrow$ a line in the plane: $y = 1 - x$



- $x + y + z = 1$

  $\rightsquigarrow$ a plane in space: $z = 1 - x - y$





- $x + y + z + w = 1$

  $\rightsquigarrow$ a "3-plane" in "4-space"...

[not pictured here]

Systems of Equations

What does the solution set of a *system* of more than one linear equation look like?

$$x\begin{pmatrix}1\\2\end{pmatrix}+y\begin{pmatrix}-3\\1\end{pmatrix}=\begin{pmatrix}-3\\8\end{pmatrix}$$

$\Downarrow$ arbitrary $\vec{b}$

Col

$$x - 3y = -3$$
$$2x + y = 8$$

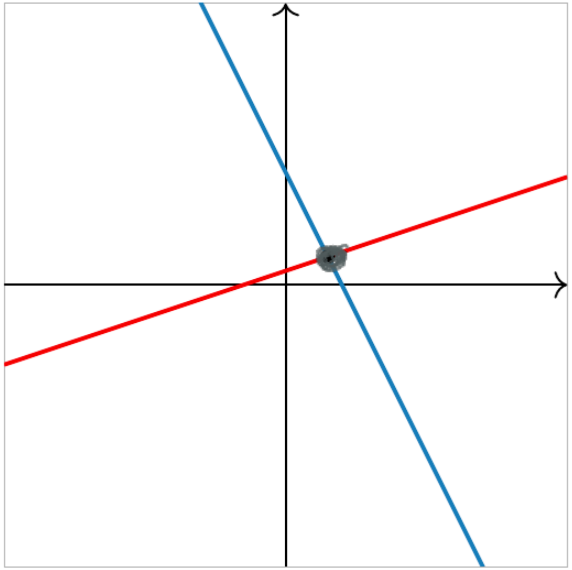
Row

$\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}-3\\1\end{pmatrix}$ are linear independent

$\Downarrow$ $\text{span}\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}-3\\1\end{pmatrix}\right\} = \mathbb{R}^2$

$\Downarrow$

The matrix $\begin{pmatrix}1 & -3\\2 & 1\end{pmatrix}$ has an inverse



…is the *intersection* of two lines, which is a *point* in this case.

In general it's an intersection of lines, planes, etc.

Systems of Equations

Column Representation

In what other ways can two lines intersect?

Matrix $\begin{pmatrix} 1 & -3 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$ $\Rightarrow \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ -3 \end{pmatrix}\right\} \neq \mathbb{R}^2$!!

$\begin{pmatrix} -3 \\ 3 \end{pmatrix} \notin \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ -3 \end{pmatrix}\right\} \Rightarrow$ No Solutions

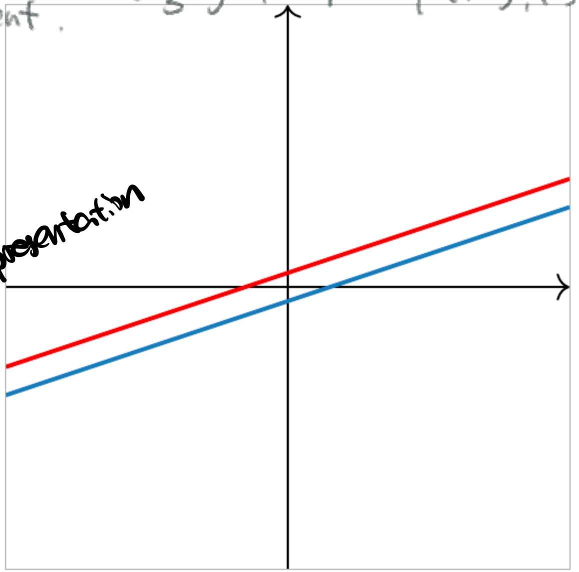
The Column Vector are linear depedent.

$$x - 3y = -3$$
$$x - 3y = 3$$

has no solution: the lines are *parallel*.

$$\begin{pmatrix} 1 & -3 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$$

Row Representation



A system of equations with no solutions is called **inconsistent**.

Systems of Equations

In what other ways can two lines intersect?

$\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}-3\\-6\end{pmatrix}$ linear dependent

$x\begin{pmatrix}1\\2\end{pmatrix} + y\begin{pmatrix}-3\\-6\end{pmatrix} = \begin{pmatrix}-3\\6\end{pmatrix}$ or $\begin{pmatrix}1 & -3\\2 & -6\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-3\\-6\end{pmatrix}$

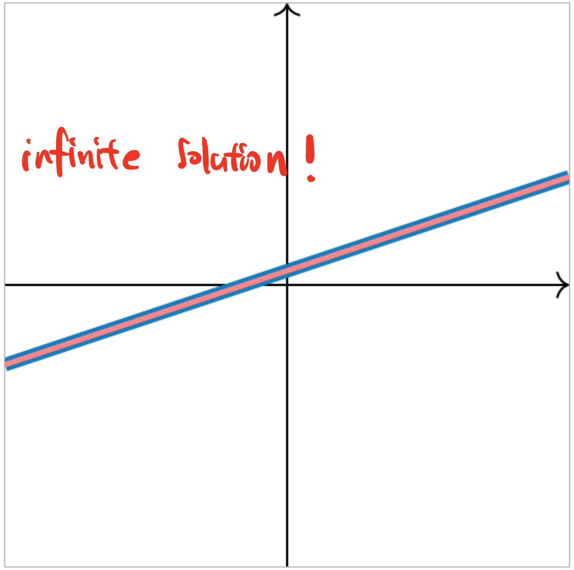
$\begin{pmatrix}-3\\-6\end{pmatrix} \in \operatorname{span}\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}-3\\-6\end{pmatrix}\right\}$

$$x - 3y = -3$$
$$2x - 6y = -6$$

col

row

has infinitely many solutions:
they are the *same line*.



Note that multiplying an equation by a nonzero number gives the *same solution set*. In other words, they are *equivalent* (systems of) equations.

Systems of Equations

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

This is the kind of problem we'll talk about for a good portion of the course.

- A **solution** is a list of numbers $x, y, z, \ldots$ that make *all* of the equations true.
- The **solution set** is the collection of all solutions.
- **Solving** the system means finding the solution set.

Systems of Equations

Consider the following system of two equations in two unknowns

$$
\begin{aligned}
x_1 - 2x_2 &= 1 \\
3x_1 + 2x_2 &= 11
\end{aligned}
$$

This system could be expressed in matrix notation as:

$$
\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}
$$

Systems of Equations – 2D – Row vs. Column Picture

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

Row picture: $(1, -2) \cdot (x_1, x_2) = 1 \implies x_1 - 2x_2 = 1$

$(3, 2) \cdot (x_1, x_2) = 11 \implies 3x_1 + 2x_2 = 11$

Systems of Equations – 2D – Row vs. Column Picture

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

Column picture: $$x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

Systems of Equations – 3D – Row vs. Column Picture

If we have three equations with three unknowns, it is still possible to draw a picture of what a solution looks like. Each of the three equations represents a plane in 3D, and their intersection gives the solution of the system. As soon as you go above 3D, visualization becomes impossible.

Consider the following system of three equations in three unknowns

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 + 5x_2 + 2x_3 &= 4 \\ 6x_1 - 3x_2 + x_3 &= 2 \end{aligned} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Systems of Equations – 3D – Row vs. Column Picture

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Row picture:

$$(1,2,3) \cdot (x_1, x_2, x_3) = 6 \quad \Longrightarrow \quad x_1 + 2x_2 + 3x_3 = 6$$

$$(2,5,2) \cdot (x_1, x_2, x_3) = 4 \quad \Longrightarrow \quad 2x_1 + 5x_2 + 2x_3 = 4$$

$$(6,-3,1) \cdot (x_1, x_2, x_3) = 2 \quad \Longrightarrow \quad 6x_1 - 3x_2 + x_3 = 2$$

Systems of Equations – 3D – Row vs. Column Picture

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Column picture: $x_1 \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$