Linear Algebra

Lecture 4

Matrix Operations

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Based on Dr. Ralph Chikhany's Slide

Reminders

- Get access to Gradescope, Campuswire.
- Obtain the textbook.
- Problem Set 1 due by 11.59 pm on Friday (NY time).
  - ✓ Late work policy applies.
- Recap Quiz 2 due by 11.59 pm on Sunday (NY time).
  - ❖ Late work policy does not apply.
- Recap Quiz is timed.
  - ❑ Once you start, you have 60 minutes to finish it (even if you close the tab)

Recap

In general, n vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly dependent if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$$

for scalars $c_1, c_2, \ldots, c_n$ not all zero.

Example: $v_1 = (1, -1, 0), v_2 = (-2, 2, 0), v_3 = (0, 0, 1)$

Recap

Linear system $Ax = b$

- have solution if and only if $b \in \text{span}\{c_1, \cdots, c_n\}$ where $c_1, \cdots, c_n$ are column vectors of matrix A

- have a unique solution if and only if matrix A have an inverse matrix A^{-1}. The unique solution is $x = A^{-1}b$. In this case, A must be a square matrix and $c_1, \cdots, c_n$ are linear independent.

Recap

Linear system $Ax = b$

- have solution if and only if $b \in \text{span}\{c_1, \cdots, c_n\}$ where $c_1, \cdots, c_n$ are column vectors of matrix A
- have a unique solution if and only if matrix A have an inverse matrix A^{-1}. The unique solution is $x = A^{-1}b$. In this case, A must be a square matrix and $c_1, \cdots, c_n$ are linear independent.

Strang Sections 2.3 – Elimination Using Matrices and 2.4 – Rules for Matrix Operations

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text

Permutation Matrices

Recall

$$I = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \xrightarrow{I\,\vec{x}} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

Permutation Matrices

$$P_{ij} = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & & & & & & \\ \color{blue}0 & \color{blue}0 & \color{blue}\dots & \color{blue}0 & \color{blue}\dots & \color{blue}1 & \color{blue}\dots & \color{blue}0 \\ \vdots & & & & & & & \\ \color{red}0 & \color{red}0 & \color{red}\dots & \color{red}1 & \color{red}\dots & \color{red}0 & \color{red}\dots & \color{red}0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

$$P_{ij} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \color{blue}x_j \\ \vdots \\ \color{red}x_i \\ \vdots \\ x_n \end{bmatrix}$$

Permutation Matrices

$$P_{ij} = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots \\ \color{blue}0 & \color{blue}0 & \color{blue}\dots & \color{blue}0 & \color{blue}\dots & \color{blue}1 & \color{blue}\dots & \color{blue}0 \\ \vdots \\ \color{red}0 & \color{red}0 & \color{red}\dots & \color{red}1 & \color{red}\dots & \color{red}0 & \color{red}\dots & \color{red}0 \\ \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

$$P_{ij} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \color{blue}x_j \\ \vdots \\ \color{red}x_i \\ \vdots \\ x_n \end{bmatrix}$$

$$P_{31} = \begin{bmatrix} \color{blue}0 & \color{blue}0 & \color{blue}1 & \color{blue}\dots & \color{blue}0 \\ 0 & 1 & 0 & \dots & 0 \\ \color{red}1 & \color{red}0 & \color{red}0 & \color{red}\dots & \color{red}0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

Permutation Matrices

$$P_{ij} = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots \\ {\color{blue}0} & {\color{blue}0} & {\color{blue}\dots} & {\color{blue}0} & {\color{blue}\dots} & {\color{blue}1} & {\color{blue}\dots} & {\color{blue}0} \\ \vdots \\ {\color{red}0} & {\color{red}0} & {\color{red}\dots} & {\color{red}1} & {\color{red}\dots} & {\color{red}0} & {\color{red}\dots} & {\color{red}0} \\ \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

$$P_{ij} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ {\color{blue}x_j} \\ \vdots \\ {\color{red}x_i} \\ \vdots \\ x_n \end{bmatrix}$$

$$P_{31} = \begin{bmatrix} {\color{blue}0} & {\color{blue}0} & {\color{blue}1} & {\color{blue}\dots} & {\color{blue}0} \\ 0 & 1 & 0 & \dots & 0 \\ {\color{red}1} & {\color{red}0} & {\color{red}0} & {\color{red}\dots} & {\color{red}0} \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \xrightarrow{{\color{red}P_{31}\,\vec{x}}} \begin{bmatrix} {\color{blue}0} & {\color{blue}0} & {\color{blue}1} & {\color{blue}\dots} & {\color{blue}0} \\ 0 & 1 & 0 & \dots & 0 \\ {\color{red}1} & {\color{red}0} & {\color{red}0} & {\color{red}\dots} & {\color{red}0} \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

Matrix Operations

Recall

Let A be an $m \times n$ matrix.

We write a_{ij} for the entry in the ith row and the jth column. It is called the ij**th entry** of the matrix.

$$\begin{array}{cc} \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix} & i\text{th row} \\ j\text{th column} & \end{array}$$

The entries $a_{11}, a_{22}, a_{33}, \ldots$ are the **diagonal entries**; they form the **main diagonal** of the matrix.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \qquad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

A **diagonal matrix** is a *square* matrix whose only nonzero entries are on the main diagonal.

$$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

The $n \times n$ **identity matrix** I_n is the diagonal matrix with all diagonal entries equal to 1. It is special because $I_n v = v$ for *all* v in $\mathbf{R}^n$.

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Recall

The **zero matrix** (of size $m \times n$) is the $m \times n$ matrix 0 with all zero entries.

$$0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix A^T whose rows are the columns of A. In other words, the ij entry of A^T is a_{ji}.

$$\underset{A}{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}} \rightsquigarrow \underset{A^T}{\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}}$$

flip

Matrix Addition and Scalar Multiplication

You add two matrices component by component, like with vectors.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{pmatrix}$$

Note you can only add two matrices *of the same size.*

You multiply a matrix by a scalar by multiplying each component, like with vectors.

$$c \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} & ca_{13} \\ ca_{21} & ca_{22} & ca_{23} \end{pmatrix}.$$

These satisfy the expected rules, like with vectors:

$$\begin{aligned} A+B &= B+A & \qquad (A+B)+C &= A+(B+C) \\ c(A+B) &= cA+cB & (c+d)A &= cA+dA \\ (cd)A &= c(dA) & A+0 &= A \end{aligned}$$

Matrix Multiplication

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ b_{21} & \cdots & b_{2m} \\ \vdots & & \\ b_{l1} & \cdots & b_{lm} \end{bmatrix} \qquad C = \begin{bmatrix} c_{11} & \cdots & c_{1k} \\ c_{21} & \cdots & c_{2k} \\ \vdots & & \\ c_{n1} & \cdots & c_{nk} \end{bmatrix}$$

Matrix Multiplication

Beware: matrix multiplication is more subtle than addition and scalar multiplication.

Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix with columns $v_1, v_2 \dots, v_p$:

must be equal

$$B = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{pmatrix}.$$

The **product** AB is the $m \times p$ matrix with columns $Av_1, Av_2, \dots, Av_p$:

The equality is a definition

$$AB \stackrel{\text{def}}{=} \begin{pmatrix} | & | & & | \\ Av_1 & Av_2 & \cdots & Av_p \\ | & | & & | \end{pmatrix}.$$

In order for $Av_1, Av_2, \dots, Av_p$ to make sense, the number of columns of A has to be the same as the number of rows of B.

Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix} = \left(\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix} \right)$$

Matrix Multiplication

A row vector of length n times a column vector of length n is a scalar:

$$\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + \cdots + a_n b_n.$$

Matrix Multiplication

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$$\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + \cdots + a_n b_n.$$

Another way of multiplying a matrix by a vector is:

$$Ax = \begin{pmatrix} \text{---} \, r_1 \, \text{---} \\ \vdots \\ \text{---} \, r_m \, \text{---} \end{pmatrix} x = \begin{pmatrix} r_1 x \\ \vdots \\ r_m x \end{pmatrix}.$$

Matrix Multiplication

A row vector of length n times a column vector of length n is a scalar:

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Another way of multiplying a matrix by a vector is:

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On the other hand, you multiply two matrices by

$$AB = A \begin{pmatrix} | & & | \\ c_1 & \cdots & c_p \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ Ac_1 & \cdots & Ac_p \\ | & & | \end{pmatrix}.$$

Matrix Multiplication

A row vector of length n times a column vector of length n is a scalar:

$$\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + \cdots + a_n b_n .$$

Another way of multiplying a matrix by a vector is:

$$Ax = \begin{pmatrix} — r_1 — \\ \vdots \\ — r_m — \end{pmatrix} x = \begin{pmatrix} r_1 x \\ \vdots \\ r_m x \end{pmatrix} .$$

On the other hand, you multiply two matrices by

$$AB = A \begin{pmatrix} | & & | \\ c_1 & \cdots & c_p \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ Ac_1 & \cdots & Ac_p \\ | & & | \end{pmatrix} .$$

It follows that

$$\boxed{AB = \begin{pmatrix} — r_1 — \\ \vdots \\ — r_m — \end{pmatrix} \begin{pmatrix} | & & | \\ c_1 & \cdots & c_p \\ | & & | \end{pmatrix} = \begin{pmatrix} r_1 c_1 & r_1 c_2 & \cdots & r_1 c_p \\ r_2 c_1 & r_2 c_2 & \cdots & r_2 c_p \\ \vdots & \vdots & & \vdots \\ r_m c_1 & r_m c_2 & \cdots & r_m c_p \end{pmatrix}}$$

Matrix Multiplication

The ij entry of $C = AB$ is the ith row of A times the jth column of B:

$$c_{ij} = (AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

This is how everybody on the planet actually computes AB. Diagram ($AB = C$):

$$
\begin{pmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ik} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mk} & \cdots & a_{mn} \end{pmatrix}_{i\text{th row}}
\cdot
\underset{j\text{th column}}{\begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{k1} & \cdots & b_{kj} & \cdots & b_{kp} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{pmatrix}}
=
\underset{ij\text{ entry}}{\begin{pmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mp} \end{pmatrix}}
$$

Matrix Multiplication

The ij entry of $C = AB$ is the ith row of A times the jth column of B:

$$c_{ij} = (AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

This is how everybody on the planet actually computes AB. Diagram ($AB = C$):

$$\begin{pmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ik} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mk} & \cdots & a_{mn} \end{pmatrix} \text{ith row} \cdot \begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{k1} & \cdots & b_{kj} & \cdots & b_{kp} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mp} \end{pmatrix}$$

jth column $\qquad$ ij entry

Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 & \square \\ \square & \square \end{pmatrix} = \begin{pmatrix} 14 & \square \\ \square & \square \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} \square & \square \\ 4 \cdot 1 + 5 \cdot 2 + 6 \cdot 3 & \square \end{pmatrix} = \begin{pmatrix} \square & \square \\ 32 & \square \end{pmatrix}$$

Matrix-Matrix and Matrix-Vector

Matrix vector multiplication is a Matrix Matrix multiplication

$$A[\vec{v}_1, \cdots, \vec{v}_k] = [A\vec{v}_1, \cdots, A\vec{v}_k]$$

$$\begin{bmatrix} \vec{r}_1^\top \\ \vec{r}_2^\top \\ \cdots \\ \vec{r}_k^\top \end{bmatrix} A = \begin{bmatrix} \vec{r}_1^\top A \\ \vec{r}_2^\top A \\ \cdots \\ \vec{r}_k^\top A \end{bmatrix}$$

Matrix Multiplication

Let $A = \begin{bmatrix} 1 & 2 & 5 & 3 \\ 3 & -1 & 0 & -3 \\ 2 & -2 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 0 \\ 4 & 1 & 3 \\ 0 & -2 & 1 \end{bmatrix}$.

Compute AB and BA (if possible).

Matrix Multiplication

Mostly matrix multiplication works like you'd expect. Suppose A has size $m \times n$, and that the other matrices below have the right size to make multiplication work.

$$\begin{aligned} A(BC) &= (AB)C & \qquad A(B+C) &= (AB+AC) \\ (B+C)A &= BA+CA & c(AB) &= (cA)B \\ c(AB) &= A(cB) & I_nA &= A \\ AI_m &= A \end{aligned}$$

Most of these are easy to verify.

Matrix Multiplication

Warnings!

- AB is usually not equal to BA.

  In fact, AB may be defined when BA is not.

- $AB = AC$ does not imply $B = C$, even if $A \neq 0$.

- $AB = 0$ does not imply $A = 0$ or $B = 0$.

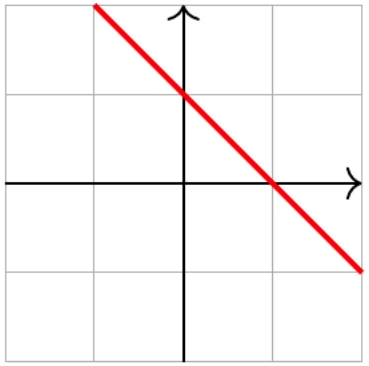
Systems of Equations

Systems of Equations

What does the solution set of a linear equation look like?

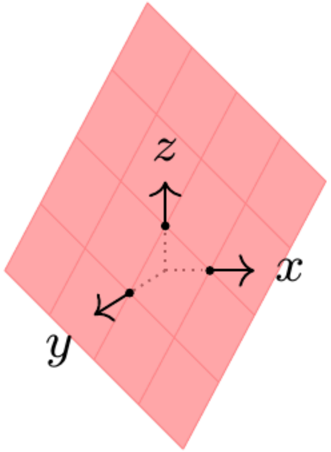
- $x + y = 1$

  $\rightsquigarrow$ a line in the plane: $y = 1 - x$



- $x + y + z = 1$

  $\rightsquigarrow$ a plane in space: $z = 1 - x - y$



- $x + y + z + w = 1$

  $\rightsquigarrow$ a "3-plane" in "4-space"...

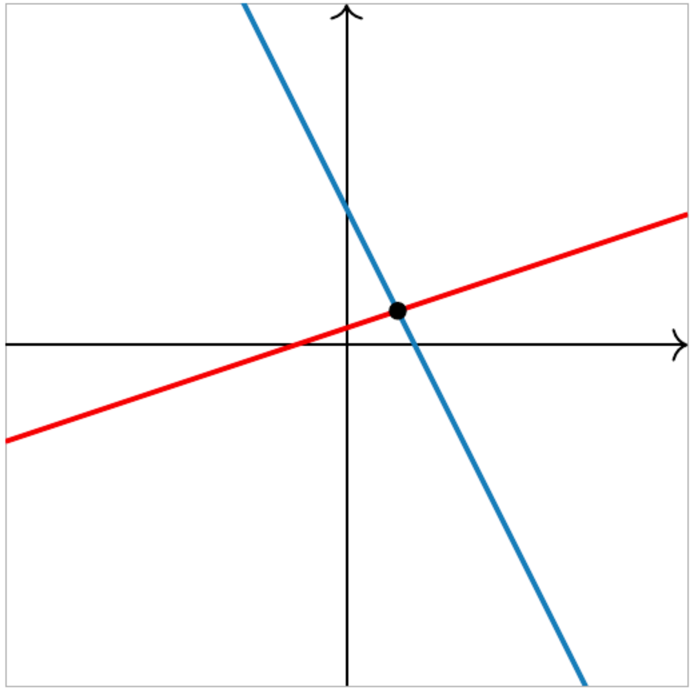
[not pictured here]

Systems of Equations

What does the solution set of a *system* of more than one linear equation look like?

$$x - 3y = -3$$
$$2x + y = 8$$

...is the *intersection* of two lines, which is a *point* in this case.



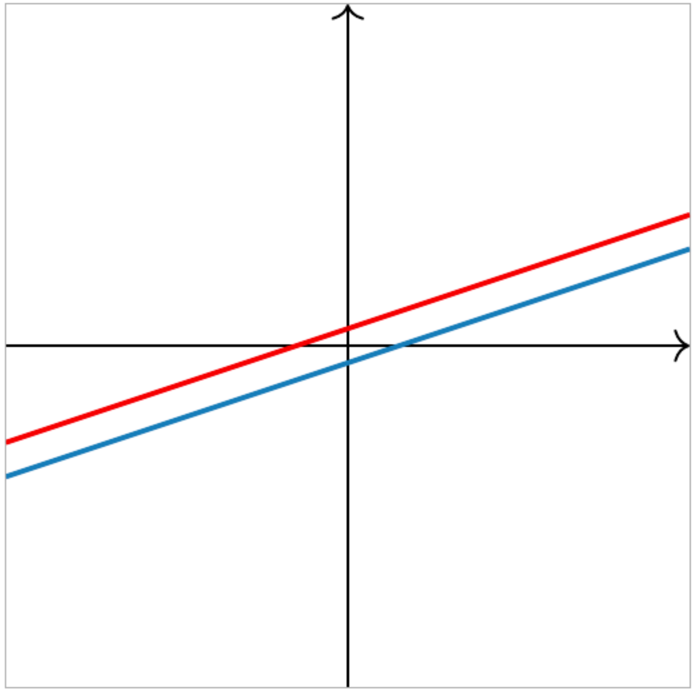
In general it's an intersection of lines, planes, etc.

Systems of Equations

In what other ways can two lines intersect?

$$x - 3y = -3$$
$$x - 3y = 3$$

has no solution: the lines are *parallel*.



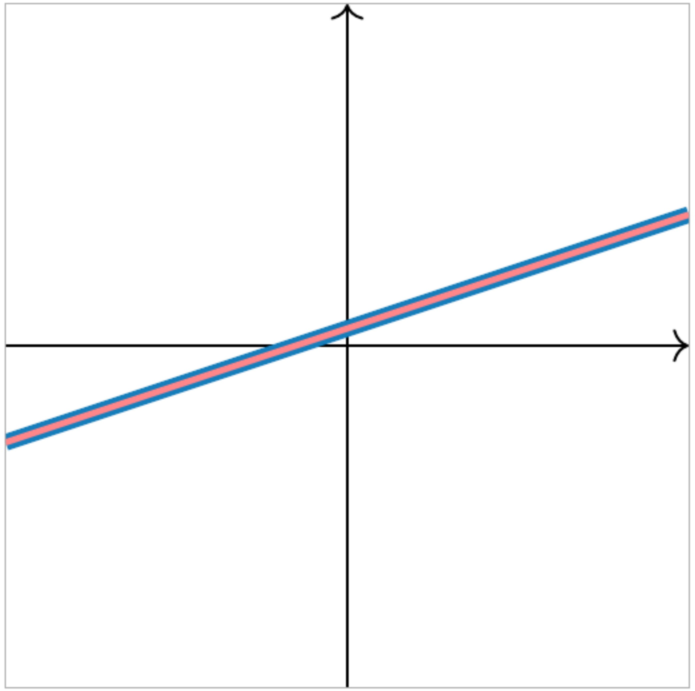
A system of equations with no solutions is called **inconsistent**.

Systems of Equations

In what other ways can two lines intersect?

$$x - 3y = -3$$
$$2x - 6y = -6$$

has infinitely many solutions:
they are the *same line*.



Note that multiplying an equation by a nonzero number gives the *same solution set*. In other words, they are *equivalent* (systems of) equations.

Systems of Equations

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

This is the kind of problem we'll talk about for a good portion of the course.

- A **solution** is a list of numbers $x, y, z, \ldots$ that make *all* of the equations true.
- The **solution set** is the collection of all solutions.
- **Solving** the system means finding the solution set.

Systems of Equations

Consider the following system of two equations in two unknowns

$$\begin{aligned} x_1 - 2x_2 &= 1 \\ 3x_1 + 2x_2 &= 11 \end{aligned}$$

This system could be expressed in matrix notation as:

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

Systems of Equations – 2D – Row vs. Column Picture

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

Row picture: $(1, -2) \cdot (x_1, x_2) = 1 \implies x_1 - 2x_2 = 1$

$(3, 2) \cdot (x_1, x_2) = 11 \implies 3x_1 + 2x_2 = 11$

Systems of Equations – 2D – Row vs. Column Picture

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

Column picture: $$x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

Systems of Equations – 3D – Row vs. Column Picture

If we have three equations with three unknowns, it is still possible to draw a picture of what a solution looks like. Each of the three equations represents a plane in 3D, and their intersection gives the solution of the system. As soon as you go above 3D, visualization becomes impossible.

Consider the following system of three equations in three unknowns

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 + 5x_2 + 2x_3 &= 4 \\ 6x_1 - 3x_2 + x_3 &= 2 \end{aligned} \implies \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Systems of Equations – 3D – Row vs. Column Picture

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Row picture:

$$\begin{aligned} (1,2,3)\cdot(x_1,x_2,x_3) &= 6 & \Longrightarrow \quad x_1 + 2x_2 + 3x_3 &= 6 \\ (2,5,2)\cdot(x_1,x_2,x_3) &= 4 & \Longrightarrow \quad 2x_1 + 5x_2 + 2x_3 &= 4 \\ (6,-3,1)\cdot(x_1,x_2,x_3) &= 2 & \Longrightarrow \quad 6x_1 - 3x_2 + x_3 &= 2 \end{aligned}$$

Systems of Equations – 3D – Row vs. Column Picture

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Column picture: $x_1 \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$