Lecture 2

Spans and Matrices

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Based on Dr. Ralph Chikhany's Slide

Reminders

- Get access to Gradescope, Campuswire.
- Obtain the textbook.
- Problem Set 1 due by 11.59 pm on Friday (NY time).
  - ✓ Late work policy applies.
- Recap Quiz 1 due by 11.59 pm on Sunday (NY time).
  - ❖ Late work policy does not apply.
- Recap Quiz is timed.
  - ❑ Once you start, you have 60 minutes to finish it (even if you close the tab)

Not every matrix have an inverse

Cyclic

$$Cx = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \boldsymbol{b}.$$

Two ways to calculate the matrix vector multiplication

Linear combination

Dot product

Not every matrix have an inverse

Cyclic

$$Cx = \underbrace{\begin{bmatrix}1 & 0 & -1\\ -1 & 1 & 0\\ 0 & -1 & 1\end{bmatrix}}_{A}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}x_1 - x_3\\ x_2 - x_1\\ x_3 - x_2\end{bmatrix} = b.$$

$b = (1,3,5)$

$$\begin{cases} x_1 - x_3 = b_1 & 1 \quad (1)\\ x_2 - x_1 = b_2 & 3 \quad (2)\\ x_3 - x_2 = b_3 & 5 \quad (3)\end{cases}$$

if Eq (1), (2) holds ⇒ Eq (3) will not hold!

add (1) + (2) + (3)

$$[x_1 - \cancel{x_3}] + [\cancel{x_2} - \cancel{x_1}] + [\cancel{x_3} - \cancel{x_2}]$$

$= 0$

$b_1 + b_2 + b_3 = 0$

Eq(1) $x_1 = x_3 + 1$

Eq(2) $x_2 = x_1 + b_2 = x_3 + 1 + 3 = x_3 + 4$

check

Eq(3) $x_3 - x_2 = x_3 - (x_3 + 4) = -4 \neq 5$

Not every matrix have an inverse

Cyclic $$Cx = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = b.$$

$$\begin{cases} \text{if } b_1+b_2+b_3 \neq 0 & \text{No solution} \\ \text{if } b_1+b_2+b_3 = 0 & \text{infinite solution} \end{cases}$$

My matrix C don't have an inverse

$b = (0,0,0)$ $b_1+b_2+b_3=0$

infinite Solution !

$$\begin{cases} x_1 - x_2 = 0 \\ x_2 - x_3 = 0 \\ x_3 - x_2 = 0 \end{cases}$$

$\Rightarrow x_1 = x_2 = x_3 =$ any real number

Recap of Column Representation

all the possible $C \cdot x$ = L.C. of the column vectors $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\Downarrow$

span of column vectors span $\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\rightarrow$ If $\vec{b}$ in the span$\{\cdots\}$ $\Leftrightarrow$ $b_1+b_2+b_3=0$

The span of column vectors is not the whole space. (3 vectors in $\mathbb{R}^3$)

Exercise

$\mathbb{R}^{3\times3}$ square matrix

$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

$$Ax = b \qquad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 + x_2 \\ x_1 - x_2 + x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

$Ax = b$

$\Downarrow$

$x = A^{-1}b$

copy A as coef

Find the solution x for any b. From $x = A^{-1}b$ read off the inverse matrix A^{-1}.

$$\begin{cases} 1\,x_1 + 0\,x_2 + 0\,x_3 = b_1 \\ -1\,x_1 + 1\,x_2 + 0\,x_3 = b_2 \\ 1\,x_1 + -1\,x_2 + 1\,x_3 = b_3 \end{cases}$$

$\longrightarrow x_1 = b_1$ only depend on x_1

$\downarrow$

know by prev Eq

$\longrightarrow x_2 = b_2 + x_1 = b_2 + b_1$ only depend on x_1 x_2

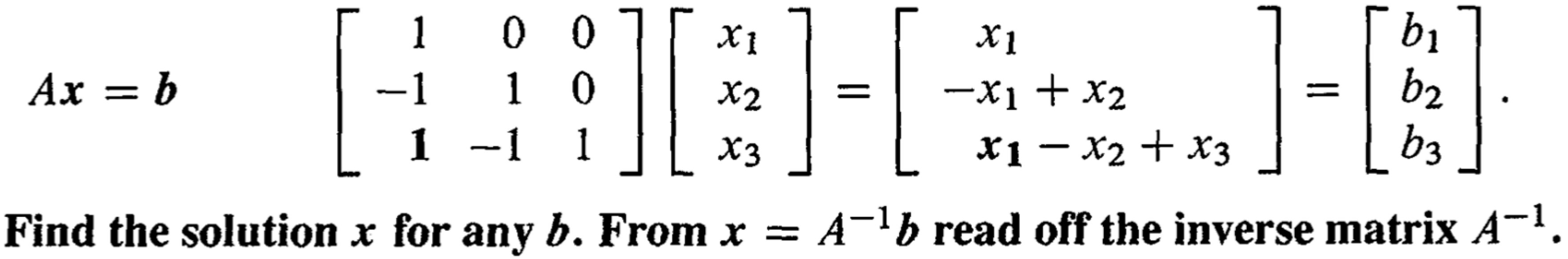
know by prev Eq

$\longrightarrow x_3 = b_3 + x_2 - x_1$ only depend on x_1 x_2 x_3

$b_3 + (b_2 + b_1) - b_1 = b_3 + b_2.$

$$\begin{cases} x_1 = b_1 \\ x_2 = b_1 + b_2 \\ x_3 = b_2 + b_3 \end{cases} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}}_{A^{-1}} \vec{b}$$

Exercise



$$Ax = b \qquad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 + x_2 \\ x_1 - x_2 + x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Find the solution x for any b. From $x = A^{-1}b$ read off the inverse matrix A^{-1}.

What is $A^{-1}b$ when $b = (0,0,1)$, $b = (0,1,0)$, $b = (1,0,0)$?

Exercise: Elimination Matrix

Later

$$E = \begin{bmatrix} 1 & 0 \\ -\ell & 1 \end{bmatrix} \qquad E^{-1} = \begin{bmatrix} 1 & 0 \\ \ell & 1 \end{bmatrix}$$

Linear Dependence and Independence

Linear In/Dependence

Sometimes the span of a set of vectors is “smaller” than you expect from the number of vectors.

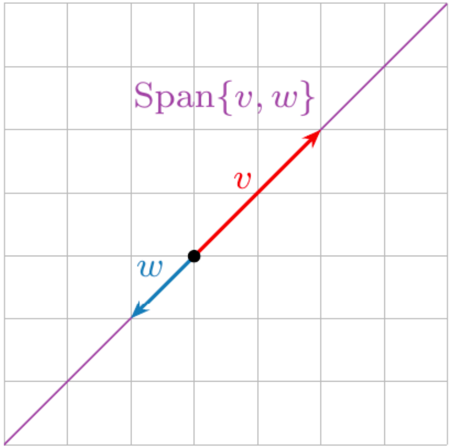
My C matrix ↓

$$C=\begin{array}{c}\begin{pmatrix}1&0&-1\\-1&1&0\\0&-1&1\end{pmatrix}\\ \quad\vec{v}\quad\ \vec{w}\quad\ \vec{u}\end{array}$$

a) 2 vectors in $\mathbb{R}^2$

if $\vec{v} \parallel \vec{w}$ (colinear), span$\{\vec{v}, \vec{w}\}$ is a line but not a plane.

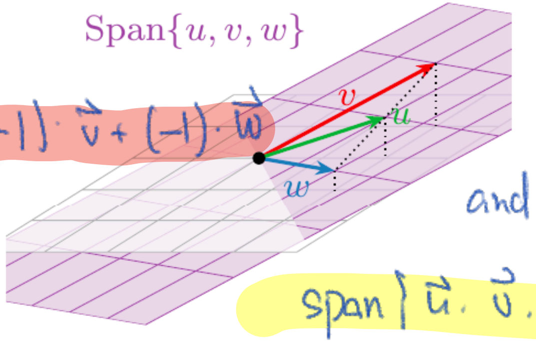
a)



b) 3 vectors in $\mathbb{R}^3$

$\vec{u}$ in span$\{\vec{v}, \vec{w}\}$

$\vec{u} = (-1)\cdot\vec{v} + (-1)\cdot\vec{w}$



① $\vec{u} = c_1\vec{v} + c_2\cdot\vec{w}$

② span$\{\vec{v}, \vec{w}\}$ is a plane and $\vec{u}$ lies on the plane

span$\{\vec{u}, \vec{v}, \vec{w}\}$ is the same plane as span$\{\vec{v}, \vec{w}\}$

This can mean many things. For example, it can mean you’re using too many vectors to write your solution set.

Notice in each case that one vector in the set is already in the span of the others—so it doesn’t make the span bigger. 1 vector → line, 2-vectors → plane, 3-vector → space

We will formalize this idea in the concept of *linear (in)dependence.* “largest” plane

Linear Dependence

linear independent : span a plane . $\vec{u} \nparallel \vec{w}$

Two vectors are said to be linearly dependent if they are multiples of each other, i.e., $\vec{u}$ and $\vec{v}$ are linearly dependent if $\vec{u} = c\vec{v}$ for some constant c.

Three vectors are linearly dependent if they all lie in the same plane, i.e., one of them is a linear combination of the other two. For example, $\vec{u}$, $\vec{v}$, and $\vec{w}$ are linearly dependent if

$\vec{u} \in \text{span}\{\vec{v}, \vec{w}\} \Rightarrow \vec{u} = c_1\vec{v} + c_2\vec{w} \Rightarrow c_1\vec{v} + c_2\vec{w} - \vec{u} = 0$

$$a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$$

$\vec{u} . \vec{v} . \vec{w}$ symmetric

for scalars a, b, and c not all zero.

In general, n vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly dependent if

L.C.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \underline{\vec{0}}$$

for scalars $c_1, c_2, \ldots, c_n$ not all zero. ✱ (at least one of $c_1 \cdots c_n$. should not be zero)

$(c_1, c_2, \cdots c_n) \neq (0, 0, \cdots, 0)$ ← trival soluti

Linear Independence

A set of n vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is said to be linearly independent if

$$\underline{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}}$$

has **only** one solution $c_1 = c_2 = \cdots = c_n = 0$. ← all the coef is zero

Yes! Linear Independent!

Example. $\vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ $\vec{w} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$

Are $\vec{u}$, $\vec{v}$, $\vec{w}$ linear independent?

$$c_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{\text{solve}}$$

Next Week

if I can find a solution $c_1 \neq 0$, or $c_2 \neq 0$ or $c_3 \neq 0$ Then They are linear dependent otherwise, ... independent.

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 + 2c_2 - c_3 = 0 \\ c_2 + 2c_3 = 0 \end{cases}$$

$$\left.\begin{array}{l} \Rightarrow c_2 = -c_1 \\ \Rightarrow c_3 = -\frac{1}{2}c_1 \end{array}\right\} \Rightarrow c_1 + 2\cdot(-c_1) - (-\tfrac{1}{2}c_1) = -\frac{1}{2}c_1 = 0 \Rightarrow c_1 = 0 \Rightarrow c_2 = c_3 = 0.$$

Combining Both

Definition

A set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ in $\mathbf{R}^n$ is **linearly independent** if the vector equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p = \vec{0}$$

has only the trivial solution $x_1 = x_2 = \cdots = x_p = 0$. The set $\{v_1, v_2, \dots, v_p\}$ is **linearly dependent** otherwise.

In other words, $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ is linearly dependent if there exist numbers $x_1, x_2, \dots, x_p$, not all equal to zero, such that

at least one - nonzero

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p = \vec{0}.$$

This is called a **linear dependence relation**.

Note that linear (in)dependence is a notion that applies to a *collection of vectors*, not to a single vector, or to one vector in the presence of some others.

Like span, linear (in)dependence is another one of those big vocabulary words that you absolutely need to learn. Much of the rest of the course will be built on these concepts, and you need to know exactly what they mean in order to be able to answer questions on quizzes and exams (and solve real-world problems later on).

An Important Result

This Lecture .

Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly *dependent* if and only if one of the vectors is in the span of the other ones.

previous lec.

The vector with the nonzero coef

Prof . This Lecture definition .

Linear Dependent $\Leftrightarrow$

$$c_1 \cdot \vec{v_1} + \cdots + c_p \vec{v_p} = \vec{0}$$

for $c_1 \cdots c_p$ at least one of them are nonzero

with loss of generality . $c_1 \neq 0$

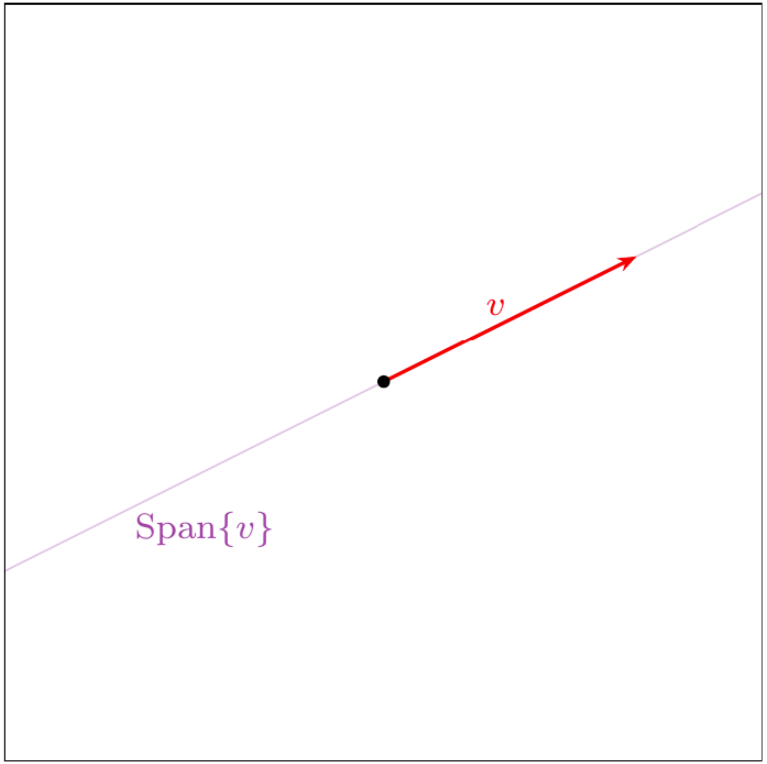
$$\Leftrightarrow \quad c_1 \cdot \vec{v_1} = -c_2 \vec{v_2} - \cdots - c_p \vec{v_p}$$

$$\Leftrightarrow \quad \vec{v_1} = -\frac{c_2}{c_1} \vec{v_2} - \cdots - \frac{c_p}{c_1} \vec{v_p}$$

$c_1 \neq 0$

$$\Leftrightarrow \quad \vec{v_1} \in \text{span}\{\vec{v_2}, \cdots, \vec{v_n}\}$$

Linear In/Dependence – Visuals in $\mathbb{R}^2$

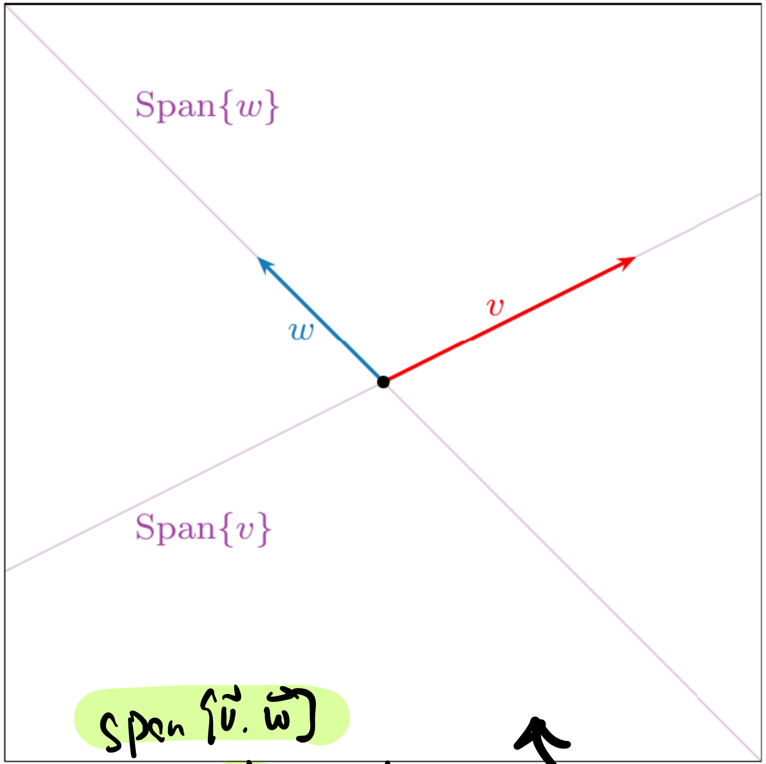


In this picture

One vector $\{v\}$:
Linearly independent if $v \neq \vec{0}$.

$c \cdot \vec{v} = \vec{0}$ Then $c = 0$

Linear In/Dependence – Visuals in $\mathbb{R}^2$



span$\{\vec{v}, \vec{w}\}$ is a plane

In this picture

One vector $\{v\}$:
Linearly independent if $v \neq 0$.

Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.

① $\vec{v}, \vec{w}$ linear dependent.

⇔ $\vec{v} \parallel \vec{w}$ (colinear) ⇔ $\vec{v} \in \text{span}\{\vec{w}\}$

$c \cdot \vec{v} = \vec{w}$

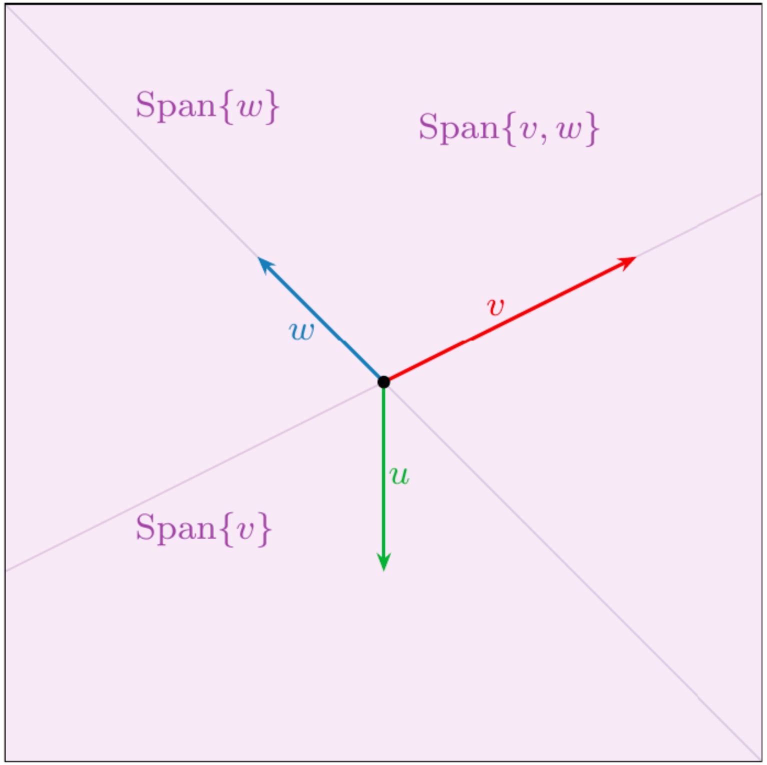
② $\vec{v}, \vec{w}$ linear independent

$\vec{v} \notin \text{span}\{\vec{w}\}$

Linear In/Dependence – Visuals in $\mathbb{R}^2$

Quiz! Can $\vec{v_1}, \vec{v_2}, \vec{v_3} \in \mathbb{R}^2$ be linear independent ?

No!



In this picture

One vector $\{v\}$:
Linearly independent if $v \neq 0$.

$\Rightarrow \{\vec{v_1}, \dots, \vec{v_k}\} \in \mathbb{R}^n$ are linear independent means $k \leq n$

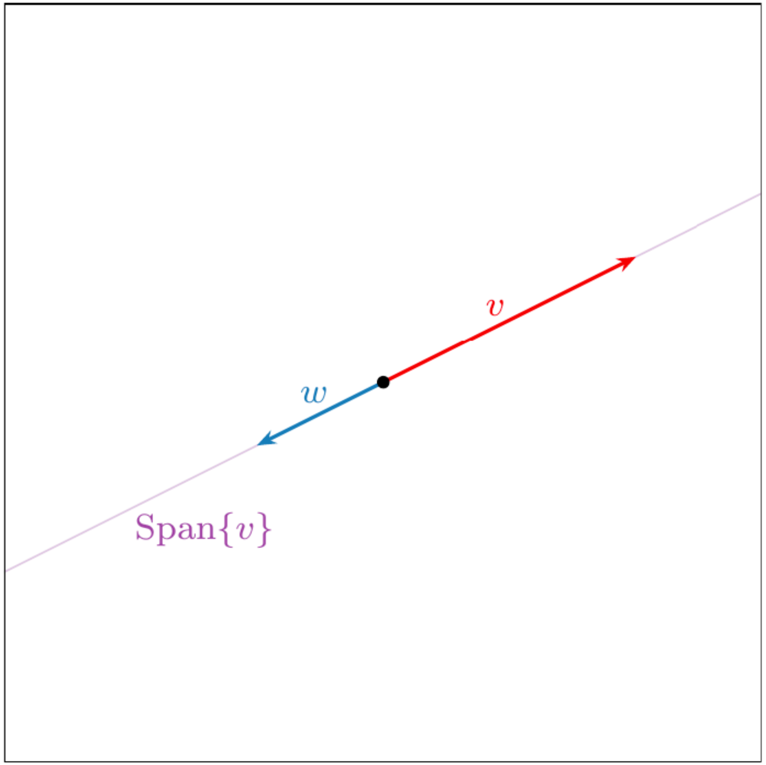
Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.

Three vectors $\{v, w, u\}$:
Linearly dependent: u is in $\mathrm{Span}\{v, w\}$.

Also v is in $\mathrm{Span}\{u, w\}$ and w is in $\mathrm{Span}\{u, v\}$.

if $\vec{u} \in \mathrm{span}\{\vec{v}, \vec{w}\} \Rightarrow \mathrm{span}\{\vec{u}, \vec{v}, \vec{w}\}$ is still a plane even I have 3- vectors.

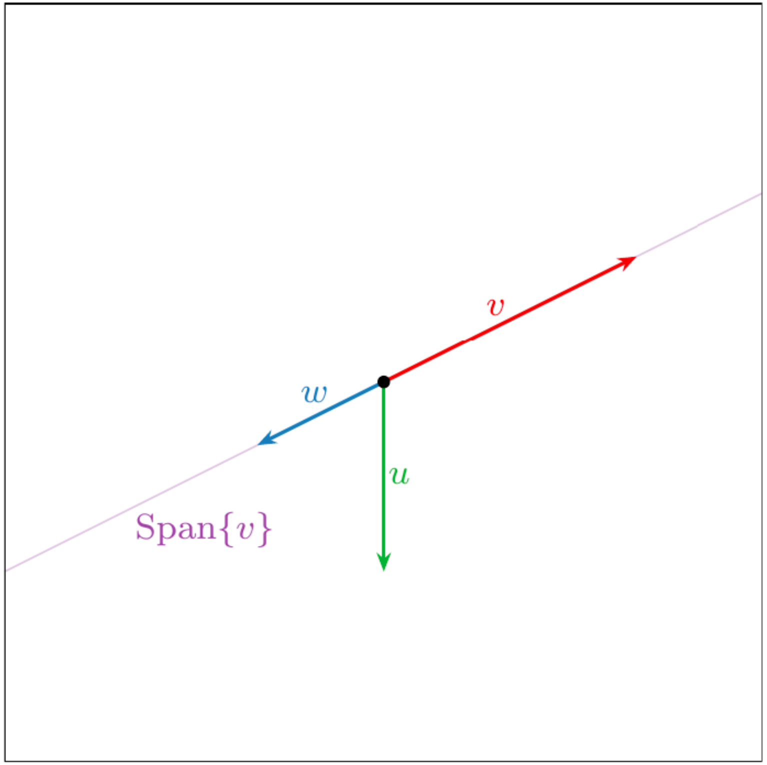
Linear In/Dependence – Visuals in $\mathbb{R}^2$



Two collinear vectors $\{v, w\}$:
Linearly dependent: w is in Span$\{v\}$ (and vice-versa).

Observe: *Two* vectors are linearly *dependent* if and only if they are *collinear*.

Linear In/Dependence – Visuals in $\mathbb{R}^2$



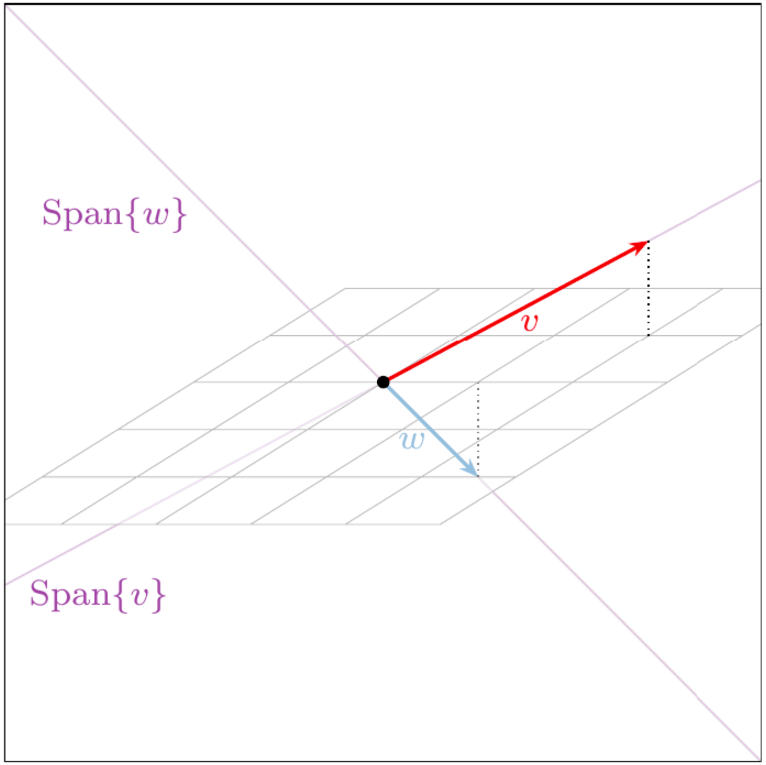
Two collinear vectors $\{v, w\}$:
Linearly dependent: w is in Span$\{v\}$ (and vice-versa).

Observe: *Two* vectors are linearly *dependent* if and only if they are *collinear*.

Three vectors $\{v, w, u\}$:
Linearly dependent: w is in Span$\{v\}$ (and vice-versa).

Observe: If a set of vectors is linearly dependent, then so is any larger set of vectors!

Linear In/Dependence – Visuals in $\mathbb{R}^3$

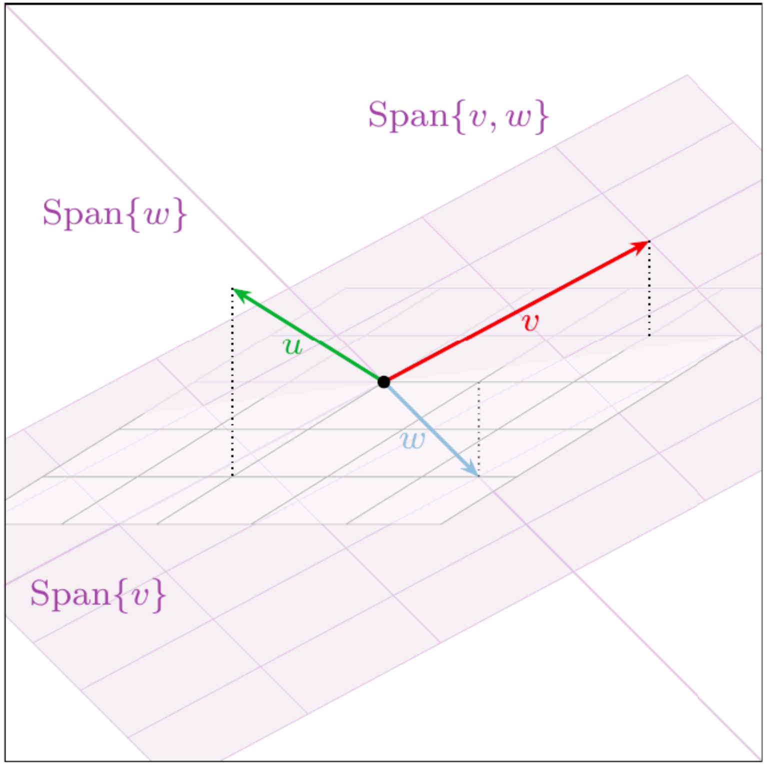


In this picture

Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.

span $\{\vec{v}, \vec{w}\}$ is plane in $\mathbb{R}^3$

Linear In/Dependence – Visuals in $\mathbb{R}^3$



In this picture

Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.

$\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}$ in $\mathbb{R}^3$

linear dependent

$\downarrow$

Three vectors $\{v, w, u\}$:
Linearly independent: no one is in the span of the other two.

if $\vec{u} \notin$ span $\{\vec{v}, \vec{w}\}$

$\Rightarrow$ linear independent.

span $\{\vec{u}, \vec{v}, \vec{w}\}$ is $\mathbb{R}^3$

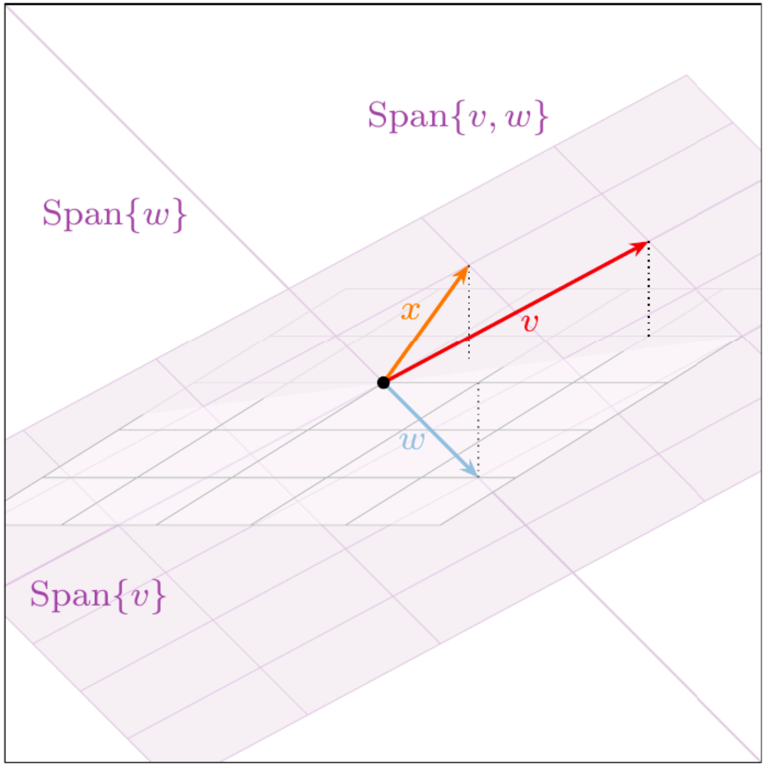
---

if $\vec{u} \in \dots$

$\Rightarrow$ linear dependent.

$\Rightarrow$ span $\{\vec{u}, \vec{v}, \vec{w}\}$ is the same place as span $\{\vec{v}, \vec{w}\}$

Linear In/Dependence – Visuals in $\mathbb{R}^3$



In this picture

Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.

Three vectors $\{v, w, x\}$:
Linearly dependent: x is in Span$\{v, w\}$.

Exercise

– "Only Square Matrix Can be Invertible" and Column Vectors should be linear Independent

Find a combination $\underline{x_1}\boldsymbol{w}_1 + \underline{x_2}\boldsymbol{w}_2 + \underline{x_3}\boldsymbol{w}_3$ that gives the zero vector:

$W_1 + W_3 = 2W_2$

$$\boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \boldsymbol{w}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \qquad \boldsymbol{w}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

Those vectors are (independent) (dependent). The three vectors lie in a <u>plane</u>. The matrix W with those columns is *not invertible*.

$$x_1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + x_3\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$W = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \Rightarrow \text{not invertible.}$$

$x_1 = 1. \quad x_3 = 1. \quad x_2 = -2$

$\vec{v}_1 \cdots \vec{v}_n$ in $\mathbb{R}^m$ and $n > m$

linear dependent!

$\| \quad W.\vec{x}$ lies in a plane

$$W = \begin{bmatrix} \underset{\mathbb{R}^m}{|} & | & | & \cdots & \underset{\mathbb{R}^m}{|} \end{bmatrix}$$

"fat" Matrix

\# col > # row

$m < n$

$$W = \begin{bmatrix} | & | & \cdots & | \end{bmatrix}$$

"Tall" matrix

\# col < # row

$\underline{m} > \underline{n}$

n Vector in $\mathbb{R}^m$

$\text{span}\{\vec{v}_1, \cdots, \vec{v}_n\}$ must smaller than $\mathbb{R}^m$

Harder example…

Not Required

If the columns combine into $A\boldsymbol{x} = \mathbf{0}$ then each row has $\boldsymbol{r} \cdot \boldsymbol{x} = 0$:

linear dependent if $\vec{x} \neq \vec{0}$

$$\begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \textbf{By rows} \quad \underbrace{\begin{bmatrix} \boldsymbol{r}_1 \cdot \boldsymbol{x} \\ \boldsymbol{r}_2 \cdot \boldsymbol{x} \\ \boldsymbol{r}_3 \cdot \boldsymbol{x} \end{bmatrix}}_{A\vec{x}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The three rows also lie in a plane. Why is that plane perpendicular to $\boldsymbol{x}$?

In Text

for square matrix

⇑

If Column Vector linear dependent

⟺ Row Vector linear dependent

NYU

Questions?