Lecture 2
Spans and Matrices

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Based on Dr. Ralph Chikhany's Slide

Reminders

- Get access to Gradescope, Campuswire.
- Obtain the textbook.
- Problem Set 1 due by 11.59 pm on Friday (NY time).
    - ✓ Late work policy applies.
- Recap Quiz 1 due by 11.59 pm on Sunday (NY time).
    - ❖ Late work policy does not apply.
- Recap Quiz is timed.
    - ❑ Once you start, you have 60 minutes to finish it (even if you close the tab)

Not every matrix have an inverse

Cyclic $$Cx = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \boldsymbol{b}.$$

Two ways to calculate the matrix vector multiplication

Linear combination

Dot product

Not every matrix have an inverse

Cyclic $$Cx = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \boldsymbol{b}.$$

$b = (1,3,5)$

Not every matrix have an inverse

Cyclic $$Cx = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \boldsymbol{b}.$$

$b = (0,0,0)$

Exercise

$$Ax = b \qquad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 + x_2 \\ x_1 - x_2 + x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Find the solution x for any b. From $x = A^{-1}b$ read off the inverse matrix A^{-1}.

Exercise

$$Ax = b \qquad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 + x_2 \\ x_1 - x_2 + x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Find the solution x for any b. From $x = A^{-1}b$ read off the inverse matrix A^{-1}.

What is $A^{-1}b$ when $b = (0,0,1)$, $b = (0,1,0)$, $b = (1,0,0)$?

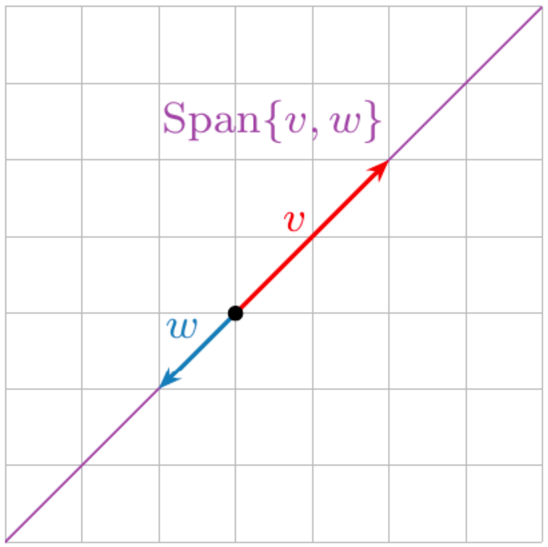
Exercise: Elimination Matrix

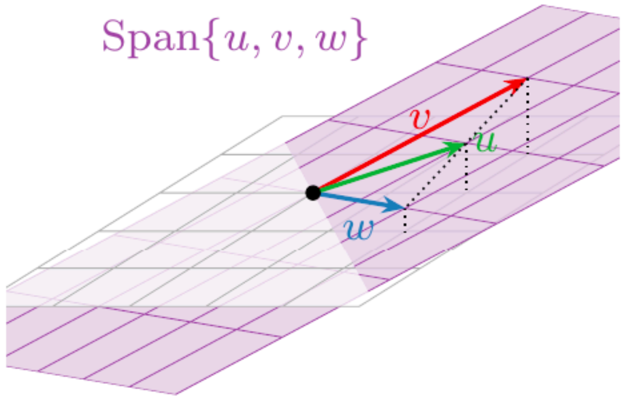
$$E = \begin{bmatrix} 1 & 0 \\ -\ell & 1 \end{bmatrix} \qquad E^{-1} = \begin{bmatrix} 1 & 0 \\ \ell & 1 \end{bmatrix}$$

Linear Dependence and Independence

Linear In/Dependence

Sometimes the span of a set of vectors is "smaller" than you expect from the number of vectors.





This can mean many things. For example, it can mean you're using too many vectors to write your solution set.

Notice in each case that one vector in the set is already in the span of the others—so it doesn't make the span bigger.

We will formalize this idea in the concept of *linear (in)dependence*.

Linear Dependence

Two vectors are said to be linearly dependent if they are multiples of each other, i.e., $\vec{u}$ and $\vec{v}$ are linearly dependent if $\vec{u} = c\vec{v}$ for some constant c.

Three vectors are linearly dependent if they all lie in the same plane, i.e., one of them is a linear combination of the other two. For example, $\vec{u}$, $\vec{v}$, and $\vec{w}$ are linearly dependent if

$$a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$$

for scalars a, b, and c not all zero.

In general, n vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly dependent if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$$

for scalars $c_1, c_2, \ldots, c_n$ not all zero.

Linear Independence

A set of n vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is said to be linearly independent if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$$

has **only** one solution $c_1 = c_2 = \cdots = c_n = 0$.

Combining Both

Definition

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ in $\mathbf{R}^n$ is **linearly independent** if the vector equation

$$x_1v_1 + x_2v_2 + \cdots + x_pv_p = 0$$

has only the trivial solution $x_1 = x_2 = \cdots = x_p = 0$. The set $\{v_1, v_2, \ldots, v_p\}$ is **linearly dependent** otherwise.

In other words, $\{v_1, v_2, \ldots, v_p\}$ is linearly dependent if there exist numbers $x_1, x_2, \ldots, x_p$, not all equal to zero, such that

$$x_1v_1 + x_2v_2 + \cdots + x_pv_p = 0.$$

This is called a **linear dependence relation**.

Note that linear (in)dependence is a notion that applies to a *collection of vectors*, not to a single vector, or to one vector in the presence of some others.

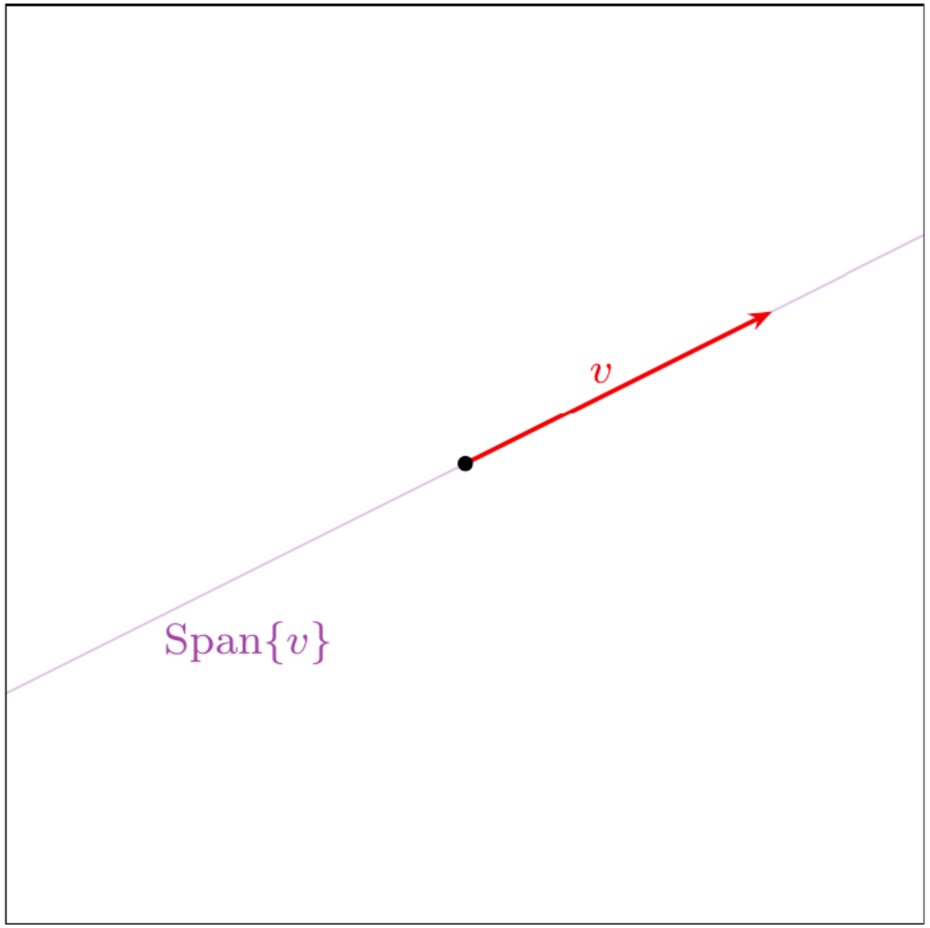
Like span, linear (in)dependence is another one of those big vocabulary words that you absolutely need to learn. Much of the rest of the course will be built on these concepts, and you need to know exactly what they mean in order to be able to answer questions on quizzes and exams (and solve real-world problems later on).

An Important Result

Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly *de*pendent if and only if one of the vectors is in the span of the other ones.

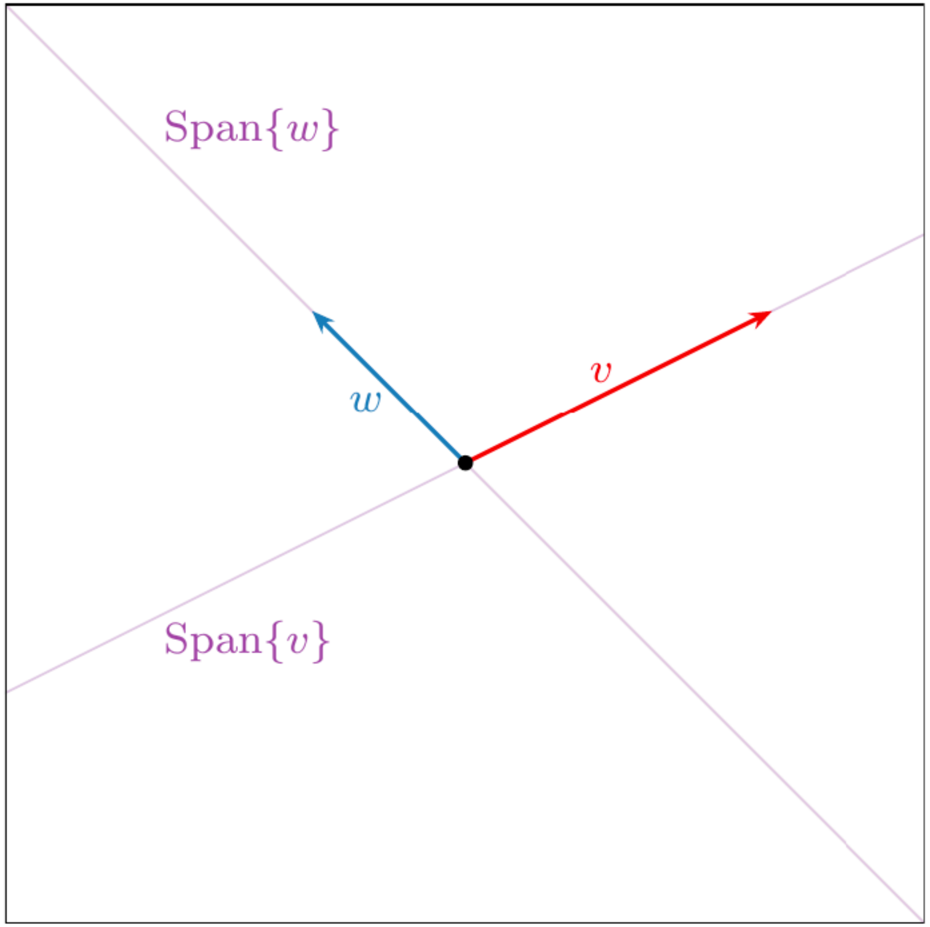
Linear In/Dependence – Visuals in $\mathbb{R}^2$



In this picture

One vector $\{v\}$:
Linearly independent if $v \neq 0$.

Linear In/Dependence – Visuals in $\mathbb{R}^2$

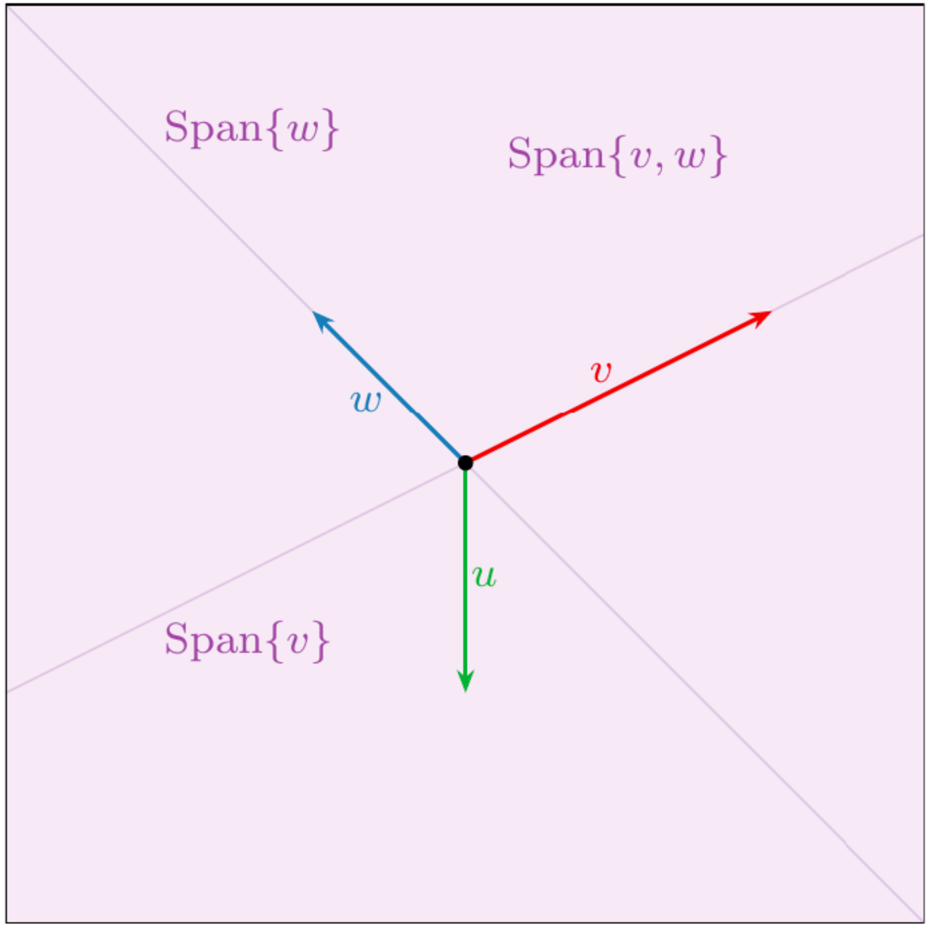


In this picture

One vector $\{v\}$:
Linearly independent if $v \neq 0$.

Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.

Linear In/Dependence – Visuals in $\mathbb{R}^2$



In this picture

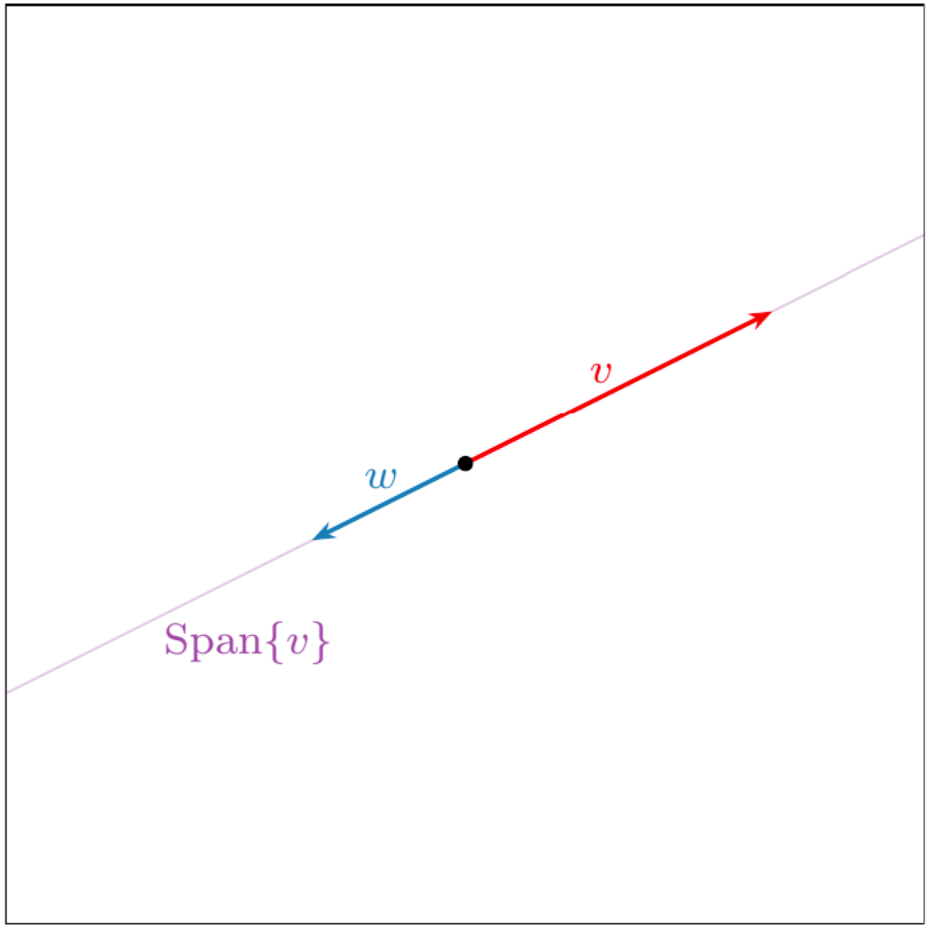
One vector $\{v\}$:
Linearly independent if $v \neq 0$.

Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.

Three vectors $\{v, w, u\}$:
Linearly dependent: u is in $\text{Span}\{v, w\}$.

Also v is in $\text{Span}\{u, w\}$ and w is in $\text{Span}\{u, v\}$.

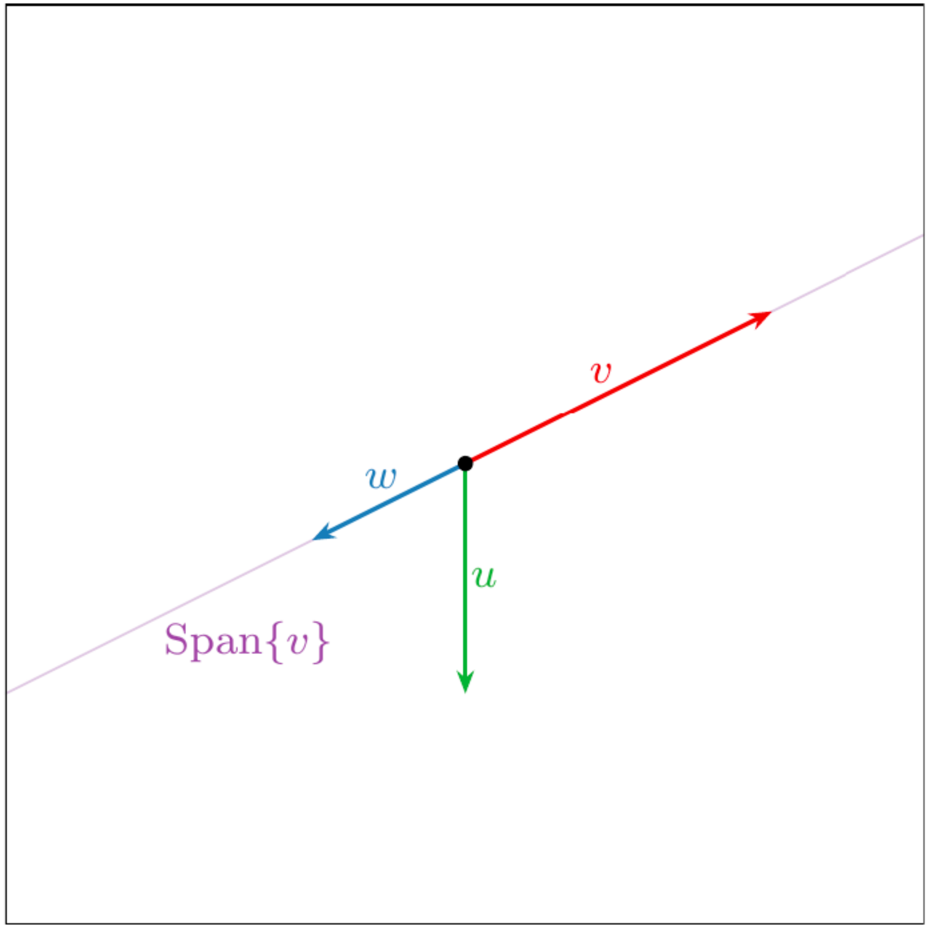
Linear In/Dependence – Visuals in $\mathbb{R}^2$



Two collinear vectors $\{v, w\}$:
Linearly dependent: w is in Span$\{v\}$ (and vice-versa).

Observe: *Two* vectors are linearly *dependent* if and only if they are *collinear*.

Linear In/Dependence – Visuals in $\mathbb{R}^2$



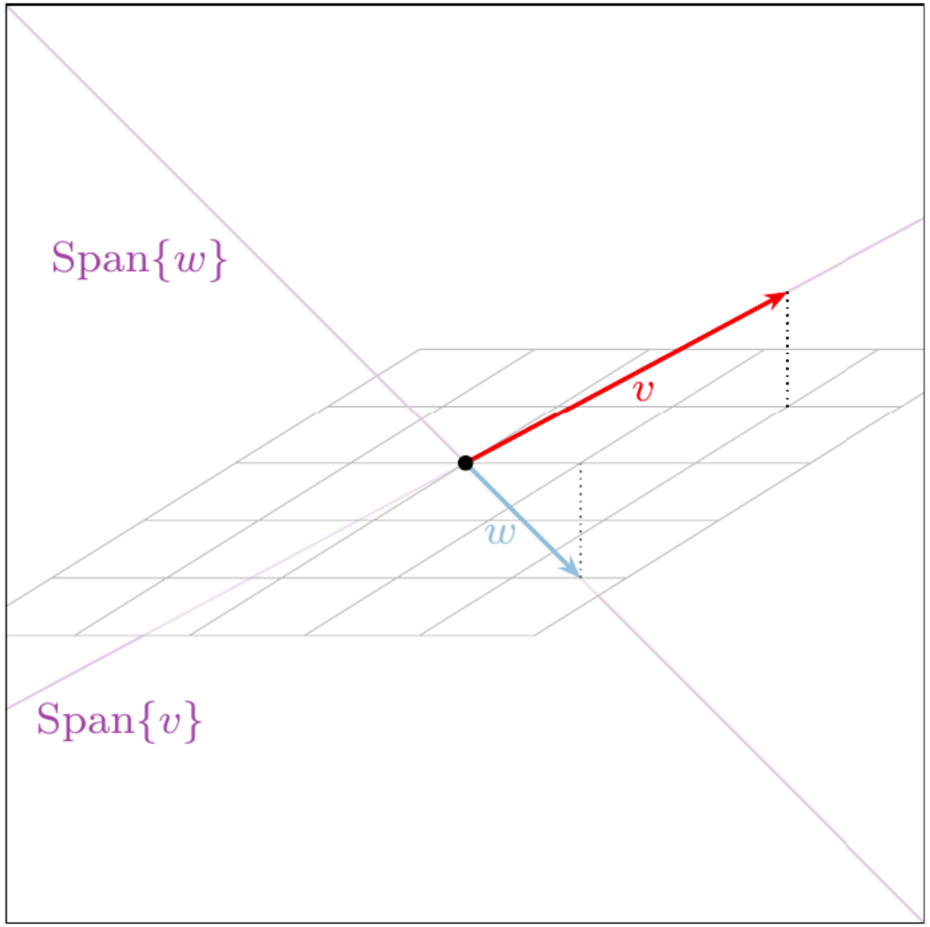
Two collinear vectors $\{v, w\}$:
Linearly dependent: w is in Span$\{v\}$ (and vice-versa).

Observe: *Two* vectors are linearly *dependent* if and only if they are *collinear*.

Three vectors $\{v, w, u\}$:
Linearly dependent: w is in Span$\{v\}$ (and vice-versa).

Observe: If a set of vectors is linearly dependent, then so is any larger set of vectors!

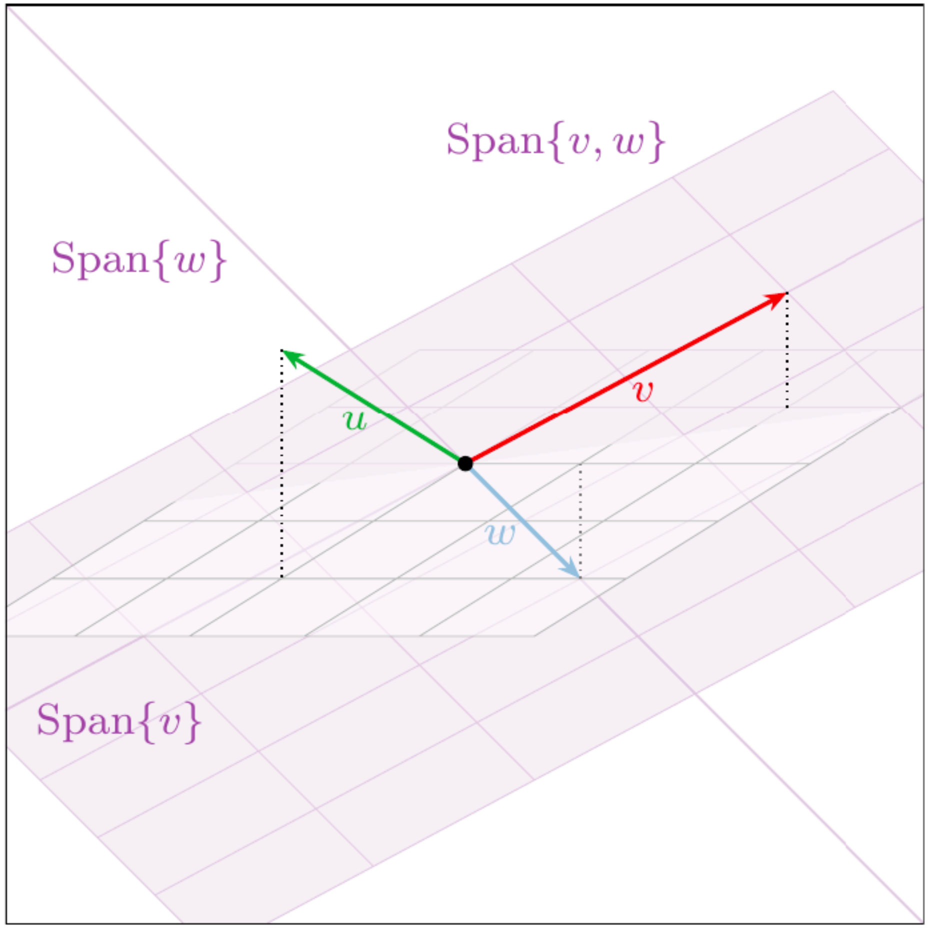
Linear In/Dependence – Visuals in $\mathbb{R}^3$



In this picture

Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.

Linear In/Dependence – Visuals in $\mathbb{R}^3$

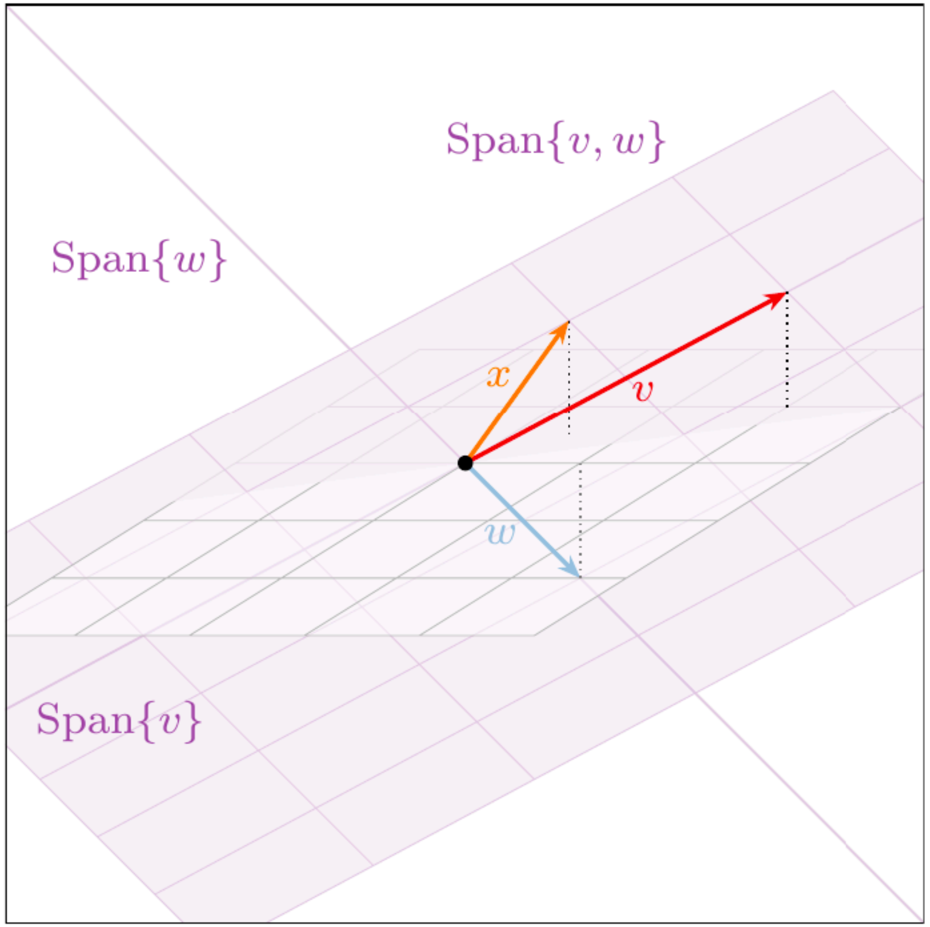


In this picture

Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.

Three vectors $\{v, w, u\}$:
Linearly independent: no one is in the span of the other two.

Linear In/Dependence – Visuals in $\mathbb{R}^3$



In this picture

Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.

Three vectors $\{v, w, x\}$:
Linearly dependent: x is in Span$\{v, w\}$.

Exercise

Find a combination $x_1\boldsymbol{w}_1 + x_2\boldsymbol{w}_2 + x_3\boldsymbol{w}_3$ that gives the zero vector:

$$\boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \boldsymbol{w}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \qquad \boldsymbol{w}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

Those vectors are (independent) (dependent). The three vectors lie in a _____. The matrix W with those columns is *not invertible*.

Harder example…

If the columns combine into $A\boldsymbol{x} = \mathbf{0}$ then each row has $\boldsymbol{r} \cdot \boldsymbol{x} = 0$:

$$\begin{bmatrix} & & \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \textbf{By rows} \begin{bmatrix} \boldsymbol{r}_1 \cdot \boldsymbol{x} \\ \boldsymbol{r}_2 \cdot \boldsymbol{x} \\ \boldsymbol{r}_3 \cdot \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The three rows also lie in a plane. Why is that plane perpendicular to $\boldsymbol{x}$?

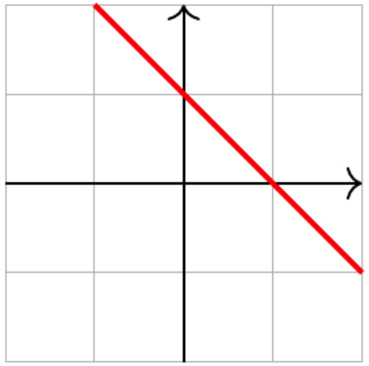
Systems of Equations

Systems of Equations

What does the solution set of a linear equation look like?

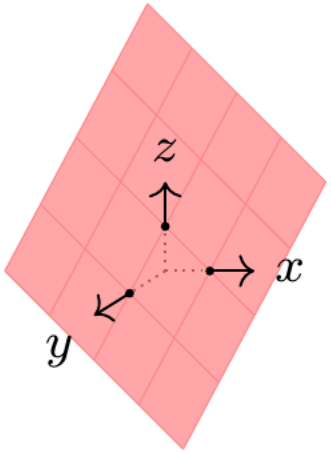
- $x + y = 1$

  $\rightsquigarrow$ a line in the plane: $y = 1 - x$



- $x + y + z = 1$

  $\rightsquigarrow$ a plane in space: $z = 1 - x - y$



- $x + y + z + w = 1$

  $\rightsquigarrow$ a "3-plane" in "4-space"...

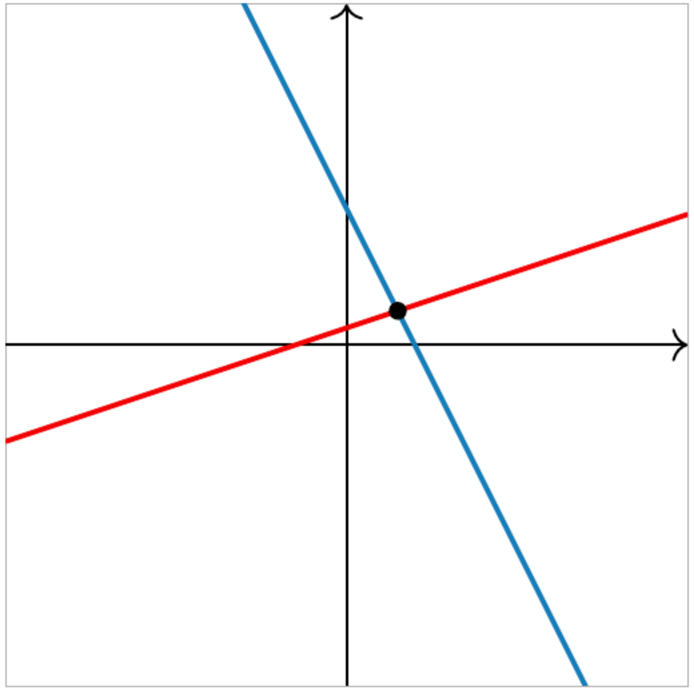
[not pictured here]

Systems of Equations

What does the solution set of a *system* of more than one linear equation look like?

$$x - 3y = -3$$
$$2x + y = 8$$

. . . is the *intersection* of two lines, which is a *point* in this case.



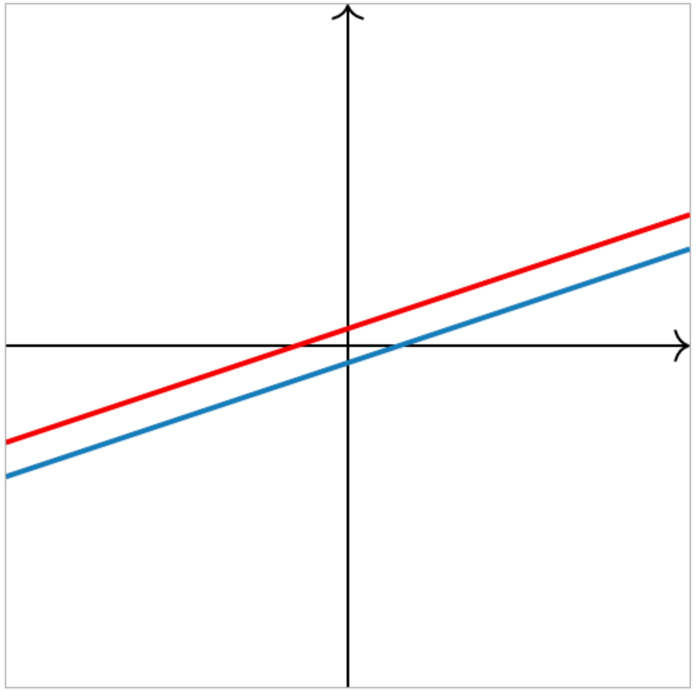
In general it's an intersection of lines, planes, etc.

Systems of Equations

In what other ways can two lines intersect?

$$x - 3y = -3$$
$$x - 3y = 3$$

has no solution: the lines are *parallel.*



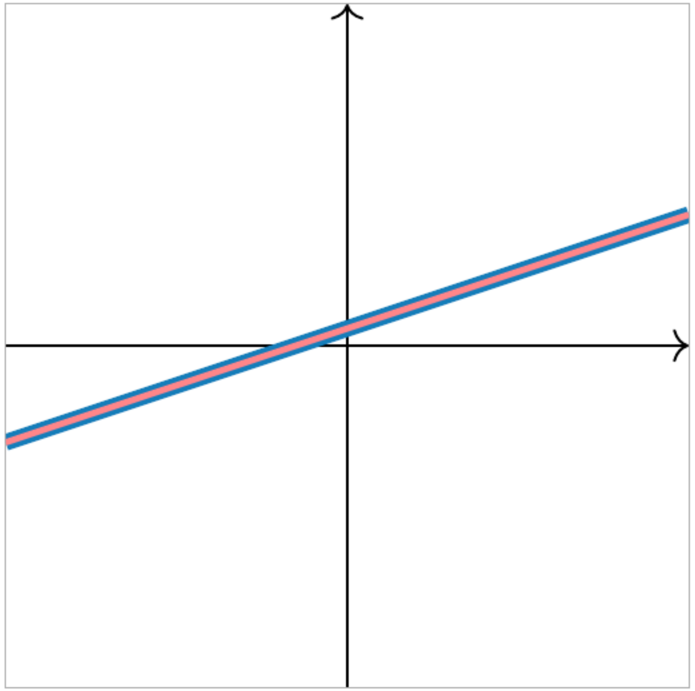
A system of equations with no solutions is called **inconsistent**.

Systems of Equations

In what other ways can two lines intersect?

$$\begin{aligned} x - 3y &= -3 \\ 2x - 6y &= -6 \end{aligned}$$

has infinitely many solutions: they are the *same line*.



Note that multiplying an equation by a nonzero number gives the *same solution set*. In other words, they are *equivalent* (systems of) equations.

Systems of Equations

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

This is the kind of problem we'll talk about for a good portion of the course.

- A **solution** is a list of numbers $x, y, z, \ldots$ that make *all* of the equations true.
- The **solution set** is the collection of all solutions.
- **Solving** the system means finding the solution set.

Systems of Equations

Consider the following system of two equations in two unknowns

$$x_1 - 2x_2 = 1$$
$$3x_1 + 2x_2 = 11$$

This system could be expressed in matrix notation as:

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

Systems of Equations – 2D – Row vs. Column Picture

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

Row picture: $(1, -2) \cdot (x_1, x_2) = 1 \implies x_1 - 2x_2 = 1$

$(3, 2) \cdot (x_1, x_2) = 11 \implies 3x_1 + 2x_2 = 11$

Systems of Equations – 2D – Row vs. Column Picture

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

Column picture: $$x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

Systems of Equations – 3D – Row vs. Column Picture

If we have three equations with three unknowns, it is still possible to draw a picture of what a solution looks like. Each of the three equations represents a plane in 3D, and their intersection gives the solution of the system. As soon as you go above 3D, visualization becomes impossible.

Consider the following system of three equations in three unknowns

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 + 5x_2 + 2x_3 &= 4 \\ 6x_1 - 3x_2 + x_3 &= 2 \end{aligned} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Systems of Equations – 3D – Row vs. Column Picture

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Row picture:

$$(1, 2, 3) \cdot (x_1, x_2, x_3) = 6 \quad \Longrightarrow \quad x_1 + 2x_2 + 3x_3 = 6$$

$$(2, 5, 2) \cdot (x_1, x_2, x_3) = 4 \quad \Longrightarrow \quad 2x_1 + 5x_2 + 2x_3 = 4$$

$$(6, -3, 1) \cdot (x_1, x_2, x_3) = 2 \quad \Longrightarrow \quad 6x_1 - 3x_2 + x_3 = 2$$

Systems of Equations – 3D – Row vs. Column Picture

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Column picture: $x_1 \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$

Questions?