NYU

Linear Algebra

Lecture 2
Spans and Matrices

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Based on Dr. Ralph Chikhany's Slide

Reminders

- Get access to Gradescope, Campuswire.
- Obtain the textbook.
- Problem Set 1 due by 11.59 pm on Friday (NY time). Next week
  - ✓ Late work policy applies.
- Recap Quiz 1 due by 11.59 pm on Sunday (NY time). This Friday on Gradescope.
  - ❖ Late work policy does not apply.
- Recap Quiz is timed.
  - ❑ Once you start, you have 60 minutes to finish it (even if you close the tab)

Cheat Sheet

- Cheat Sheet overleaf: https://www.overleaf.com/read/jjbswyyqvzdx#8803d5

Recap

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be a set of vectors in $\mathbb{R}^n$. We define

$$\operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\} = \text{set of all linear combinations of } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$$

Recap

Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ be a set of vectors in $\mathbb{R}^n$. We define

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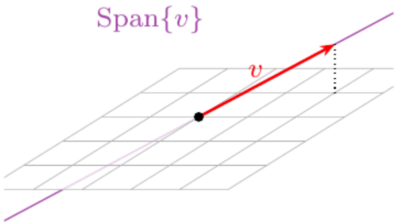
$u, v, w \in \mathbb{R}^3$



Recap

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be a set of vectors in $\mathbb{R}^n$. We define

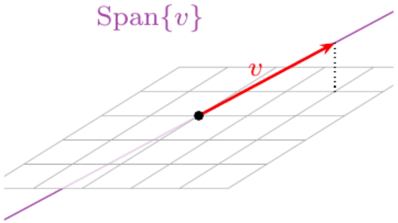
$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\} = \text{set of all linear combinations of } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$$

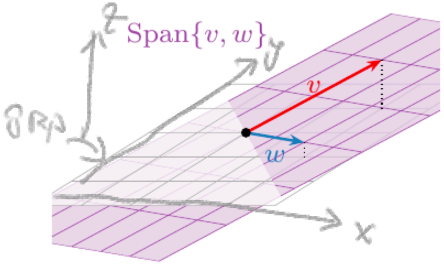
$\vec{w} = c \cdot \vec{v}$ in quiz

$\vec{w}$ is not parallel / colinear with $\vec{v}$

Then span$\{\vec{v}, \vec{w}\}$ is a plane in $\mathbb{R}^3$

case 1





← purple plane here is the span

case 2

$\vec{w}$ is colinear / parallel with $\vec{v}$

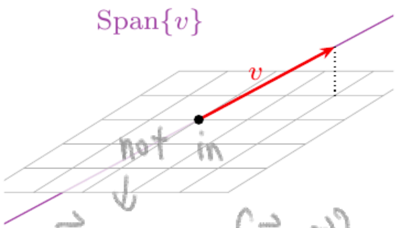
span$\{\vec{v}, \vec{w}\}$ = span$\{\vec{v}\}$ is still a line.

Recap

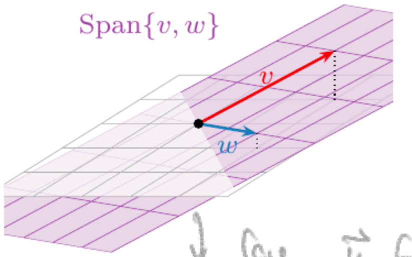
Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be a set of vectors in $\mathbb{R}^n$. We define

$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\} = \text{set of all linear combinations of } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$$

2-Vector



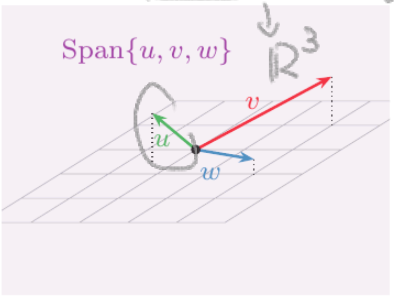
not in

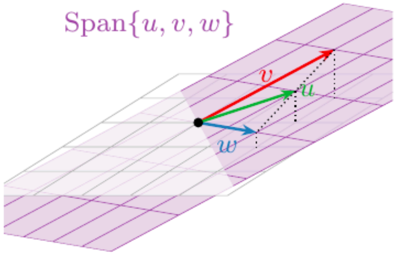


3- Vector

Case1 $\vec{u} \notin$ span$\{\vec{w}, \vec{v}\}$ $\to \mathbb{R}^3$

Case. $\vec{u} \in$ span$\{\vec{v}, \vec{v}\}$





my span will not change

span$\{\vec{u}, \vec{v}, \vec{w}\}$ = span$\{\vec{v}, \vec{w}\}$

Strang Section 1.3 - Matrices

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text

Matrices

An $m \times n$ matrix A is a rectangular array of (real) numbers a_{ij} with m rows and n columns, where

m: # rows

n: # columns

$m \times n$ matrix as $\mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

a_{ij} → column index (j)

↓ row index (i)

A matrix is called **square** if it is $n \times n$, i.e., it has the same number of rows and columns.

square $\mathbb{R}^{n \times n}$

Example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$ is 2×3 matrix $A \in \mathbb{R}^{2 \times 3}$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

$a_{12} = 2$

$a_{23} = 6$

Matrices

Let A be an $m \times n$ matrix.

We write a_{ij} for the entry in the ith row and the jth column. It is called the ij**th entry** of the matrix.

$$\begin{array}{cc} \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix} & i\text{th row} \\ j\text{th column} & \end{array}$$

The entries $a_{11}, a_{22}, a_{33}, \ldots$ are the **diagonal entries**; they form the **main diagonal** of the matrix.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \qquad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

diag elements of matrix are a_{ii} $i = 1, \ldots, \min\{m, n\}$

the smaller one of m an n

A **diagonal matrix** is a *square* matrix whose only nonzero entries are on the main diagonal.

$$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

a_{12} a_{31}

← diag matrix $\in \mathbb{R}^{n \times n}$

only a_{ii} can be non-zero

$a_{ij} = 0$ if $i \neq j$

Example $a_{12} = 0$ $1 \neq 2$

$a_{13} = 0$ $1 \neq 3$

The $n \times n$ **identity matrix** I_n is the diagonal matrix with all diagonal entries equal to 1. It is special because $I_n v = v$ for *all* v in $\mathbf{R}^n$.

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

← later

Matrices

is not zero

The **zero matrix** (of size $m \times n$) is the $m \times n$ matrix 0 with all zero entries.

$$0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix A^T whose rows are the columns of A. In other words, the ij entry of A^T is a_{ji}.

$$\overset{A}{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}} \rightsquigarrow \overset{A^T}{\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}}$$

flip

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots \\ a_{21} & & & & \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots \\ \cdots & & & \vdots & \cdots \end{pmatrix} \quad \text{i-th row}$$

i-th colum   j-th column.

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{i1} & \cdots \\ a_{12} & & & \vdots & \\ \vdots & & & \vdots & \\ a_{1j} & \cdots & & a_{ij} & \cdots \\ \vdots & & & \vdots & \end{pmatrix} \quad \text{j-th row}$$

j-th row

i-th colum

<u>Example</u>

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$\Rightarrow A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

<u>Example</u> $A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is a vector $\mathbb{R}^{3\times 1}$ 3-row 1-column

$A^T = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ $\mathbb{R}^{1\times 3}$ 1-row 3-column

Column vs. Row Representation of a Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

vector 1 vector 2 vector n

Example

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \in \mathbb{R}^{2\times 3}$

$\vec{v}_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \in \mathbb{R}^2$

$\vec{v}_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

$\vec{v}_3 = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

$A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$

each column of A is a vector $\in \mathbb{R}^m$

$\vec{v}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \cdots \vec{v}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$

Then $A = [\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n]$

Column vs. Row Representation of a Matrix

Example $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \in \mathbb{R}^{2\times 3}$

$\vec{r}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\vec{r}_1^{\,T} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

$\vec{r}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ $\vec{r}_2^{\,T} = \begin{pmatrix} 4 & 5 & 6 \end{pmatrix}$

$A = \begin{bmatrix} \vec{r}_1^{\,T} \\ \vec{r}_2^{\,T} \end{bmatrix}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$[a_{11} \cdots a_{1j} \cdots a_{1n}] = \begin{bmatrix} a_{11} \\ \vdots \\ a_{1n} \end{bmatrix}^T$

$\vec{r}_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{1n} \end{pmatrix} \in \mathbb{R}^n$

$r_2 = \begin{pmatrix} a_{21} \\ \vdots \\ a_{2n} \end{pmatrix} \in \mathbb{R}^n$

m- row vector

each row vector $\in \mathbb{R}^n$

Then $A = \begin{bmatrix} r_1^T \\ r_2^T \\ \vdots \\ \vec{r}_m \end{bmatrix}$

Linear Combination in Matrix Notation

Multiply a matrix with a vector

- Column Understanding
- Row Understanding

A linear combination of n vectors, $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, in $\mathbb{R}^m$ is given by

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n$$

a vector in $\mathbb{R}^m$

↑ will give

where $x_1, x_2, \ldots, x_n \in \mathbb{R}$.

$A \in \mathbb{R}^{m\times n}$ can only multiply with a vector $\vec{x} \in \mathbb{R}^n$

This can be expressed as an $m \times n$ matrix A multiplying a vector $\vec{x} \in \mathbb{R}^n$

$$\underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x} \in \mathbb{R}^n} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n$$

$\vec{v}_1 \in \mathbb{R}^m$, …, $\vec{v}_n$

$\mathbb{R}^m$ $\in \mathbb{R}^m$ $\mathbb{R}^m$

a vector in $\mathbb{R}^m$

!! n is the # row of vector

You can understand a vector in $\mathbb{R}^m$ as a $\mathbb{R}^{m\times 1}$ matrix

Pool

$\{Ax \mid x \in \mathbb{R}^n\}$ gives me the span.

$A \in \mathbb{R}^{m \times n}$ $\quad x \in \mathbb{R}^n$

$A = [\vec{v}_1 \cdots \cdots \vec{v}_n]$

Ax lie in the span of the column vectors of matrix A

$Ax \in \mathbb{R}^m$

$\underline{Ax \in \text{span}\{\vec{v}_1 \cdots \vec{v}_n\}}$ Yes!

is L.C. of $\vec{v}_1 \cdots \vec{v}_n$

What is the size of matrix A

For all the vector v ($\in \mathbb{R}^m$) in the span of the column vectors of matrix A, we can find a vector x ($\in \mathbb{R}^n$), such that $Ax = v$ Yes!

if $\vec{v} \in \text{span}\{\vec{v}_1 \cdots \vec{v}_n\}$

can we find a vector x s.t. $Ax = \vec{v}$

$\vec{v}$ is L.C. of $\{\vec{v}_1 \cdots \vec{v}_n\}$ $\xrightarrow{\text{means}}$ $\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$. Then $A\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{v}$

Dot product as matrix vector multiplication

$x \cdot y$ is $x^\top y$

! here is a dot

$\vec{x} \cdot \vec{y} = x^T y$

$$x \cdot y = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \stackrel{\text{def}}{=} x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

→ is a real number.

understand $\vec{x} \cdot \vec{y}$ as a L.C.

$x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$.

weights / coef.

Transpose of $\vec{y}$

Then $[y_1 \cdots y_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$\underline{\underline{\text{by defn}}}$ $x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$.

a vector in $\mathbb{R}^m$ is a $\mathbb{R}^{m\times 1}$ matrix

$\vec{y} \cdot \vec{x} = y^T x \in \mathbb{R}^n$

$\vec{y} \in \mathbb{R}^n$, $\vec{x} \in \mathbb{R}^n$, $y \in \mathbb{R}^{n\times 1}$, $y^T \in \underline{\underline{\mathbb{R}^{1\times n}}}$

→ $\mathbb{R}^1$ is real number

$\mathbb{R}^{m\times n}$ can multiply $\mathbb{R}^n$

identity matrix

The $n \times n$ **identity matrix** I_n is the diagonal matrix with all diagonal entries equal to 1. It is special because $I_n v = v$ for *all* v in $\mathbf{R}^n$.

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

← all other are zero

identity Matrix

main diag element are 1

$$I_3 \cdot \vec{x} = \vec{x}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

① $I_3 = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} \end{bmatrix}$

$$\vec{v_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{v_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \vec{v_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$I_3 \vec{x} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \vec{x}$$

② Row

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_3 \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} (1,0,0) \cdot (x_1, x_2, x_3) \\ (0,1,0) \cdot (x_1, x_2, x_3) \\ (0,0,1) \cdot (x_1, x_2, x_3) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Linear Combination in Matrix Notation

Example: Let $A = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \\ \begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array} \end{array}$, and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Compute $A\vec{x}$.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A\vec{x} = x_1 \cdot \vec{v}_1 + x_2 \cdot \vec{v}_2 + x_3 \cdot \vec{v}_3$$

$$= x_1 \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + 0 \cdot x_2 + 0 \cdot x_3 \\ -x_1 + x_2 + 0 \cdot x_3 \\ 0 \cdot x_1 - x_2 + x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix}$$

Dot Product with Rows

They are the same for

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1,0,0) \cdot (x_1,x_2,x_3) \\ (-1,1,0) \cdot (x_1,x_2,x_3) \\ (0,-1,1) \cdot (x_1,x_2,x_3) \end{bmatrix}.$$

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1,0,0) \cdot (x_1,x_2,x_3) \\ (-1,1,0) \cdot (x_1,x_2,x_3) \\ (0,-1,1) \cdot (x_1,x_2,x_3) \end{bmatrix}.$$

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1,0,0) \cdot (x_1,x_2,x_3) \\ (-1,1,0) \cdot (x_1,x_2,x_3) \\ (0,-1,1) \cdot (x_1,x_2,x_3) \end{bmatrix}.$$

Matrix times vector

Matrix times vector $$Ax = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} \begin{bmatrix} c \\ d \\ e \end{bmatrix} = c\vec{v}_1 + d\vec{v}_2 + e\vec{v}_3. \qquad (3)$$

$\vec{x}$

Dot Product View:

$$A\vec{x} = \begin{bmatrix} \vec{r}_1^{\,T} \\ \vec{r}_2^{\,T} \\ \vec{r}_m^{\,T} \end{bmatrix} \cdot \vec{x} = \begin{bmatrix} \vec{r}_1 \cdot \vec{x} \\ \vec{r}_2 \cdot \vec{x} \\ \vdots \\ \vec{r}_m \cdot \vec{x} \end{bmatrix}$$

← easier one to do calculation.

m-row

each row do a dot product → $\mathbb{R}^m$ vector

$A \in \mathbb{R}^{m \times n}$

Then $\vec{r}_i \in \mathbb{R}^n$, $\vec{x} \in \mathbb{R}^n$ The same size ← They can do dot product.

$A \in \mathbb{R}^{m \times n} \quad x \in \mathbb{R}^n \rightarrow Ax \in \mathbb{R}^m$

Examples

Let v_1, v_2, v_3 be vectors in $\mathbf{R}^3$. How can you write the vector equation

coef

$$2v_1 + 3v_2 - 4v_3 = \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}$$

in terms of matrix multiplication?

use the vector here as column vector

$$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$$

$$x = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 1 \end{bmatrix}$$

The system $Ax = b$

matrix $\mathbb{R}^{m\times n}$ vector $\cancel{\mathbb{R}}^n$ $\mathbb{R}^m$

The result of $A\vec{x}$, where A is an $m \times n$ matrix and $\vec{x} \in \mathbb{R}^n$ is a vector $\vec{b} \in \mathbb{R}^m$, where

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

coef unknown

Then $A\vec{x} = \vec{b}$ ← right hand side

by defn $\parallel$

$x_1\vec{v}_1 + \cdots + x_n\vec{v}_n$ $\Updownarrow$

If A is a square matrix, i.e., A is $n \times n$, and $\vec{x} \in \mathbb{R}^n$, then $A\vec{x} = \vec{b} \in \mathbb{R}^n$.

$\begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}$ $\begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix}$

Linear System $\Longleftrightarrow$ L.C.

$b_1 = x_1 \cdot a_{11} + x_2\, a_{12} + \cdots + x_n \cdot a_{1n}$

$b_2 = x_1 \cdot a_{21} + x_2\, a_{22} + \cdots + x_n \cdot a_{2n}$

$\vdots$

$b_m = x_1 \cdot a_{m1} + x_2\, a_{m2} + \cdots + x_n\, a_{mn}$

if $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$ is L.C. of $\vec{v}_1 \ldots \vec{v}_n$

if $\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = x_1 \cdot \vec{v}_1 + \cdots + x_n \vec{v}_n$

The system $Ax = b$: What if x is unknown?

The result of $A\vec{x}$, where A is an $m \times n$ matrix and $\vec{x} \in \mathbb{R}^n$ is a vector $\vec{b} \in \mathbb{R}^m$, where

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

When A and $\vec{x}$ are given, computing $\vec{b}$ is straight forward. However, the reverse is not always true (or even possible). That is, if A and $\vec{b}$ are given , it is not always possible to find $\vec{x}$.

If A is a square matrix, i.e., A is $n \times n$, and $\vec{x} \in \mathbb{R}^n$, then $A\vec{x} = \vec{b} \in \mathbb{R}^n$.

Next lectures, is to solve $\underset{\text{coef}}{A}\,\underset{\text{unknown}}{\vec{x}} = \underset{\text{right hand side}}{\vec{b}}$

$$\begin{cases} a_{11}x_1 & & & = b_1 \\ & a_{22}x_2 & & = b_2 \\ & & \ddots & \vdots \\ & & a_{nn}x_n & = b_n \end{cases}$$

Why diag matrix (square) is easy

$$A = \begin{pmatrix} a_{11} & & \\ & \ddots & \\ & & a_{nn} \end{pmatrix}$$

are zero

$Ax = b$

$$\begin{cases} a_{11} \cdot x_1 + a_{12}x_2 \cdots + a_{1n}x_n = b_1 \\ a_{21} \cdot x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

are zero

Examples

Consider the system $A\vec{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \vec{b}.$

← are zero

Suppose that b_1, b_2, and b_3 are given, and you want to compute x_1, x_2, and x_3 in terms of the components of $\vec{b}$.

upper Triangular part are zero.

"lower Traingular"

Coef

right hand side.

$$\begin{cases} x_1 = b_1 \\ -x_1 + x_2 = b_2 \\ -x_2 + x_3 = b_3 \end{cases}$$

→ $x_1 = b_1$

→ $x_2 = b_1 + b_2$

→ $x_3 = b_1 + b_2 + b_3$.

easy to solve because. we can solve. one by one.

Examples

Consider the system $A\vec{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \vec{b}$.

Suppose that b_1, b_2, and b_3 are given, and you want to compute x_1, x_2, and x_3 in terms of the components of $\vec{b}$.

inverse of matrix

Solution

$$\vec{x} = A^{-1}\vec{b}$$

Equation

$$A\vec{x} = \vec{b}$$

coef unknown Right hand side

$$\vec{x} = B\vec{b}$$

$$\begin{aligned} x_1 &= b_1 \\ -x_1 + x_2 &= b_2 \\ -x_2 + x_3 &= b_3 \end{aligned}$$

Solution $\begin{aligned} x_1 &= b_1 \\ x_2 &= b_1 + b_2 \\ x_3 &= b_1 + b_2 + b_3. \end{aligned}$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Not every matrix have an inverse

Cyclic $$Cx = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = b.$$

Two ways to calculate the matrix vector multiplication

Linear combination

Next time.

"Not every matrix have a inverse"

Dot product

Not every matrix have an inverse

Cyclic $$Cx = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = b.$$

$b = (1,3,5)$

Not every matrix have an inverse

Cyclic $$C\boldsymbol{x} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \boldsymbol{b}.$$

$b = (0,0,0)$

Review

- Two way to calculate the Matrix-vector multiplication

Row Representation

$A = \begin{bmatrix} - \vec{r}_1^T - \\ \vdots \\ \vec{r}_m^T \end{bmatrix} \in \mathbb{R}^{m\times n}$

$\vec{r}_i \in \mathbb{R}^n$ $\quad \vec{x} \in \mathbb{R}^n$

$Ax = \begin{bmatrix} \vec{r}_1 \cdot \vec{x} \\ \vdots \\ \vec{r}_m \cdot \vec{x} \end{bmatrix} \in \mathbb{R}^m$

Linear System $\begin{cases} 2x + y = 1 \\ 3x + 2y = 1 \end{cases}$

Column Representation

$A = [\vec{v}_1 \ \cdots \ \vec{v}_n] \in \mathbb{R}^{m\times n}$

$\vec{v}_i \in \mathbb{R}^m$

$Ax = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n$ is a LC. $\in \mathbb{R}^m$

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$

$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ unknown R.H.S.

Questions?