

Lecture 2  
**Spans and Matrices**

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Based on Dr. Ralph Chikhany's Slide

# Reminders

- Get access to Gradescope, Campuswire.
- Obtain the textbook.
- Problem Set 1 due by 11.59 pm on Friday (NY time).
  - ✓ Late work policy applies.
- Recap Quiz 1 due by 11.59 pm on Sunday (NY time).
  - ❖ Late work policy does not apply.
- Recap Quiz is timed.
  - ☐ Once you start, you have 60 minutes to finish it (even if you close the tab)

# Cheat Sheet

- Cheat Sheet overleaf: <https://www.overleaf.com/read/jjbswyyqvzdx#8803d5>

# Recap

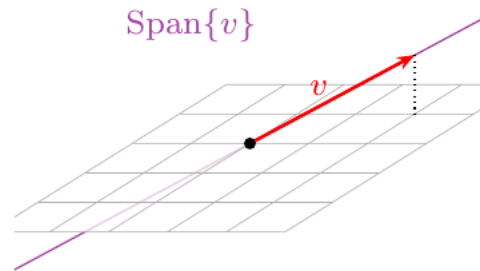
Let  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$  be a set of vectors in  $\mathbb{R}^n$ . We define

$\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\} = \text{set of all linear combinations of } \vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$

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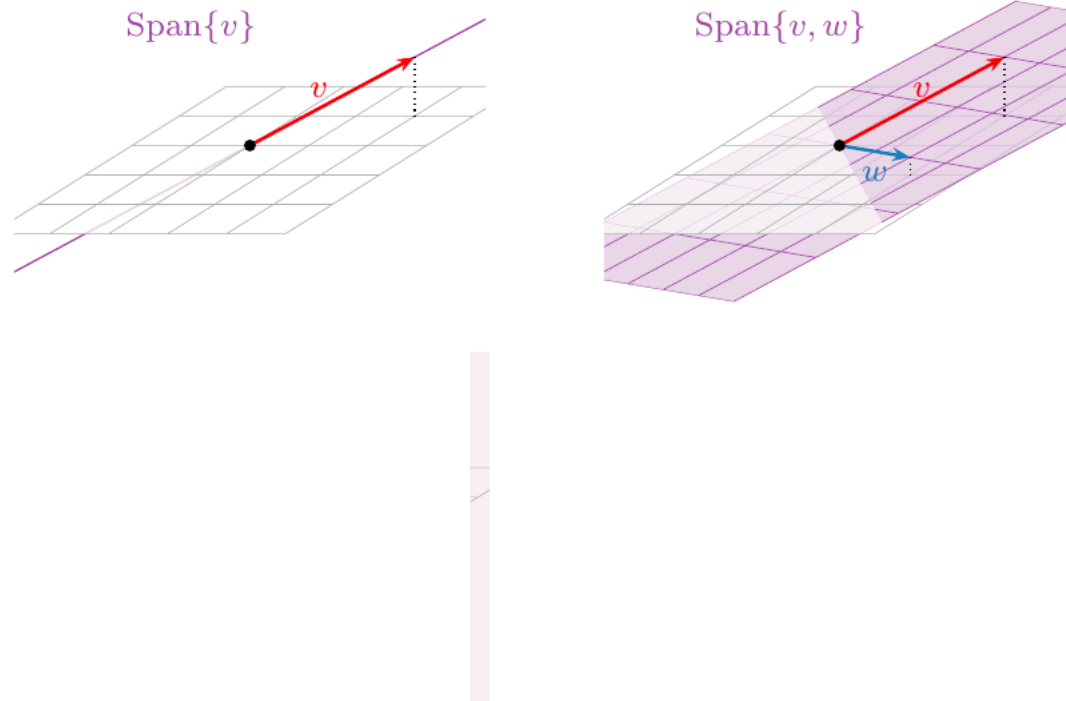
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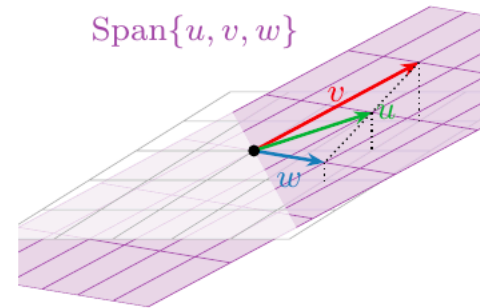
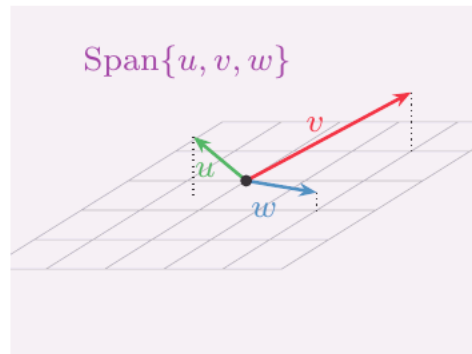
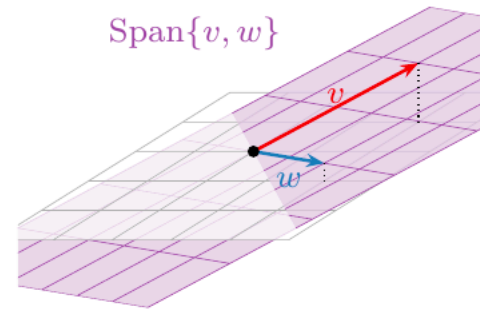
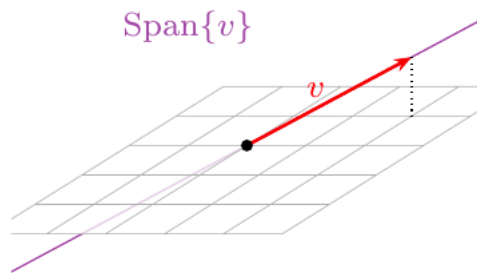
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## Strang Section 1.3 - Matrices

Course notes adapted from *Introduction to Linear Algebra* by Strang (5<sup>th</sup> ed),  
N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by  
Margalit and Rabinoff, in addition to our text



# Matrices

An  $m \times n$  matrix  $A$  is a rectangular array of (real) numbers  $a_{ij}$  with  $m$  rows and  $n$  columns, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

A matrix is called **square** if it is  $n \times n$ , i.e., it has the same number of rows and columns.

# Matrices

Let  $A$  be an  $m \times n$  matrix.

We write  $a_{ij}$  for the entry in the  $i$ th row and the  $j$ th column. It is called the  **$ij$ th entry** of the matrix.

$$\begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix}$$

$j$ th column

$i$ th row

The entries  $a_{11}, a_{22}, a_{33}, \dots$  are the **diagonal entries**; they form the **main diagonal** of the matrix.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

A **diagonal matrix** is a *square* matrix whose only nonzero entries are on the main diagonal.

$$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

The  $n \times n$  **identity matrix**  $I_n$  is the diagonal matrix with all diagonal entries equal to 1. It is special because  $I_n v = v$  for all  $v$  in  $\mathbf{R}^n$ .

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

# Matrices

The **zero matrix** (of size  $m \times n$ ) is the  $m \times n$  matrix  $0$  with all zero entries.

$$0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The **transpose** of an  $m \times n$  matrix  $A$  is the  $n \times m$  matrix  $A^T$  whose rows are the columns of  $A$ . In other words, the  $ij$  entry of  $A^T$  is  $a_{ji}$ .

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \rightsquigarrow A^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}$$

# Column vs. Row Representation of a Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

# Column vs. Row Representation of a Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

# Linear Combination in Matrix Notation

A linear combination of  $n$  vectors,  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ , in  $\mathbb{R}^m$  is given by

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n$$

where  $x_1, x_2, \dots, x_n \in \mathbb{R}$ .

This can be expressed as an  $m \times n$  matrix  $A$  multiplying a vector  $\vec{x} \in \mathbb{R}^n$

$$\underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{\vec{x}}$$

# Pool

$Ax$  lie in the span of the column vectors of matrix  $A$

What is the size of matrix  $A$

For all the vector  $v$  in the span of the column vectors of matrix  $A$ , we can find a vector  $x$ , such that  $Ax = v$

# Dot product as matrix vector multiplication

$x \cdot y$  is  $x^\top y$

$$x \cdot y = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \stackrel{\text{def}}{=} x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$



# identity matrix

The  $n \times n$  **identity matrix**  $I_n$  is the diagonal matrix with all diagonal entries equal to 1. It is special because  $I_n v = v$  for *all*  $v$  in  $\mathbf{R}^n$ .

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

# Linear Combination in Matrix Notation

**Example:** Let  $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ , and  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ . Compute  $A\vec{x}$ .

# Dot Product with Rows

$$A\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1, 0, 0) \cdot (x_1, x_2, x_3) \\ (-1, 1, 0) \cdot (x_1, x_2, x_3) \\ (0, -1, 1) \cdot (x_1, x_2, x_3) \end{bmatrix}.$$

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$$A\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1, 0, 0) \cdot (x_1, x_2, x_3) \\ (-1, 1, 0) \cdot (x_1, x_2, x_3) \\ (0, -1, 1) \cdot (x_1, x_2, x_3) \end{bmatrix}.$$

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# Matrix times vector

**Matrix times vector**

$$Ax = \begin{bmatrix} u & v & w \end{bmatrix} \begin{bmatrix} c \\ d \\ e \end{bmatrix} = cu + dv + ew. \quad (3)$$

**Dot Product View:**

# Examples

Let  $v_1, v_2, v_3$  be vectors in  $\mathbf{R}^3$ . How can you write the vector equation

$$2v_1 + 3v_2 - 4v_3 = \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}$$

in terms of matrix multiplication?

# The system $A\mathbf{x} = \mathbf{b}$

The result of  $A\vec{x}$ , where  $A$  is an  $m \times n$  matrix and  $\vec{x} \in \mathbb{R}^n$  is a vector  $\vec{b} \in \mathbb{R}^m$ , where

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

If  $A$  is a square matrix, i.e.,  $A$  is  $n \times n$ , and  $\vec{x} \in \mathbb{R}^n$ , then  $A\vec{x} = \vec{b} \in \mathbb{R}^n$ .

# The system $A\mathbf{x} = \mathbf{b}$ : What if $\mathbf{x}$ is unknown?

The result of  $A\vec{x}$ , where  $A$  is an  $m \times n$  matrix and  $\vec{x} \in \mathbb{R}^n$  is a vector  $\vec{b} \in \mathbb{R}^m$ , where

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

When  $A$  and  $\vec{x}$  are given, computing  $\vec{b}$  is straight forward. However, the reverse is not always true (or even possible). That is, if  $A$  and  $\vec{b}$  are given, it is not always possible to find  $\vec{x}$ .

If  $A$  is a square matrix, i.e.,  $A$  is  $n \times n$ , and  $\vec{x} \in \mathbb{R}^n$ , then  $A\vec{x} = \vec{b} \in \mathbb{R}^n$ .

# Examples

Consider the system  $A\vec{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \vec{b}.$

Suppose that  $b_1$ ,  $b_2$ , and  $b_3$  are given, and you want to compute  $x_1$ ,  $x_2$ , and  $x_3$  in terms of the components of  $\vec{b}$ .



# Examples

Consider the system  $A\vec{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \vec{b}.$

Suppose that  $b_1$ ,  $b_2$ , and  $b_3$  are given, and you want to compute  $x_1$ ,  $x_2$ , and  $x_3$  in terms of the components of  $\vec{b}$ .

$$x_1 = b_1$$

$$-x_1 + x_2 = b_2$$

$$-x_2 + x_3 = b_3$$

$$x_1 = b_1$$

**Solution**  $x_2 = b_1 + b_2$

$$x_3 = b_1 + b_2 + b_3.$$

# Not every matrix have an inverse

**Cyclic**

$$C\mathbf{x} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \mathbf{b}.$$

Two ways to calculate the matrix vector multiplication

Linear combination

Dot product

# Not every matrix have an inverse

**Cyclic**

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$$\mathbf{b} = (1, 3, 5)$$

# Not every matrix have an inverse

**Cyclic**

$$C\mathbf{x} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \mathbf{b}.$$

$$\mathbf{b} = (0,0,0)$$

# Review

- Two way to calculate the Matrix-vector multiplication

Row Representation

Column Representation



Questions?