Lecture 22

Nonstandard Bases and Change of Bases

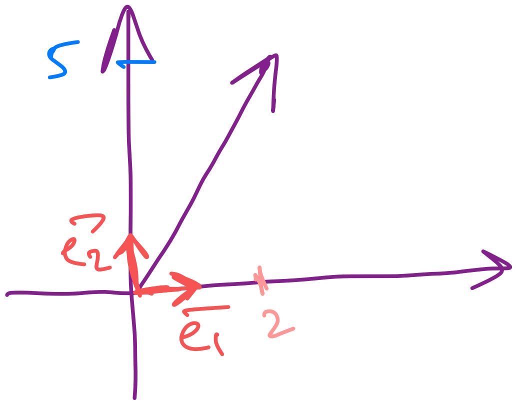
Dr. Ralph Chikhany

Strang Section 8.2 – The Matrix of a Linear Transformation and Section 8.3 – The Search for a Good Basis

Course notes adapted from N. Hammoud's NYU lecture notes.

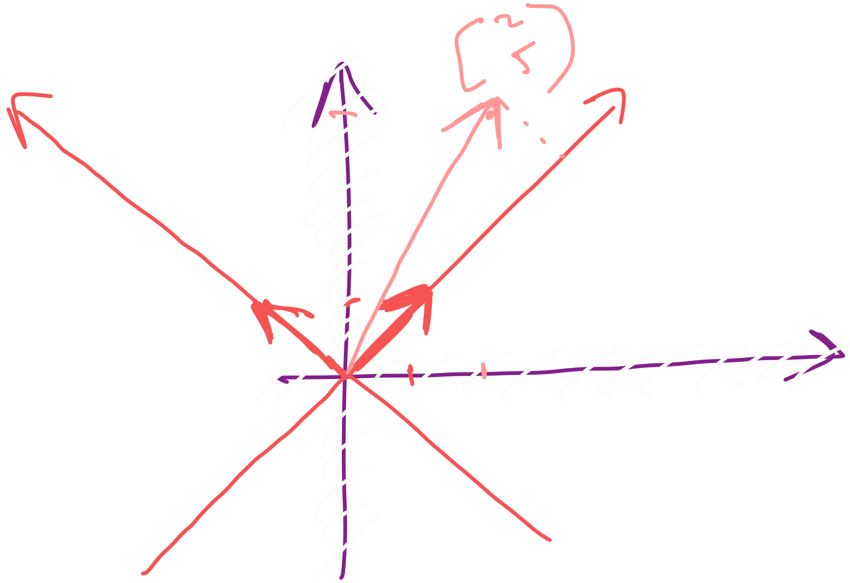
Intro

Consider the vector $\begin{bmatrix} 2 \\ 5 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \underbrace{2\vec{e_1} + 5\vec{e_2}}$



$$\mathcal{B}_{\mathbb{R}^2} = \{\vec{e_1}, \vec{e_2}\}$$

But what if $\mathcal{B}'_{\mathbb{R}^2} = \{\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{v_1}}, \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\vec{v_2}}\}$?



Need the coordinates of $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ in terms of the new basis elements

old: $\begin{bmatrix} 2 \\ 5 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

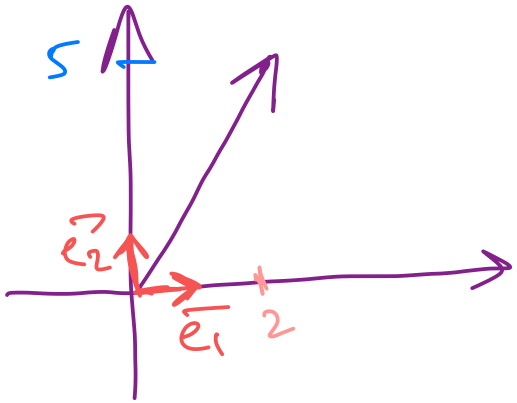
new: $\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Find α and β

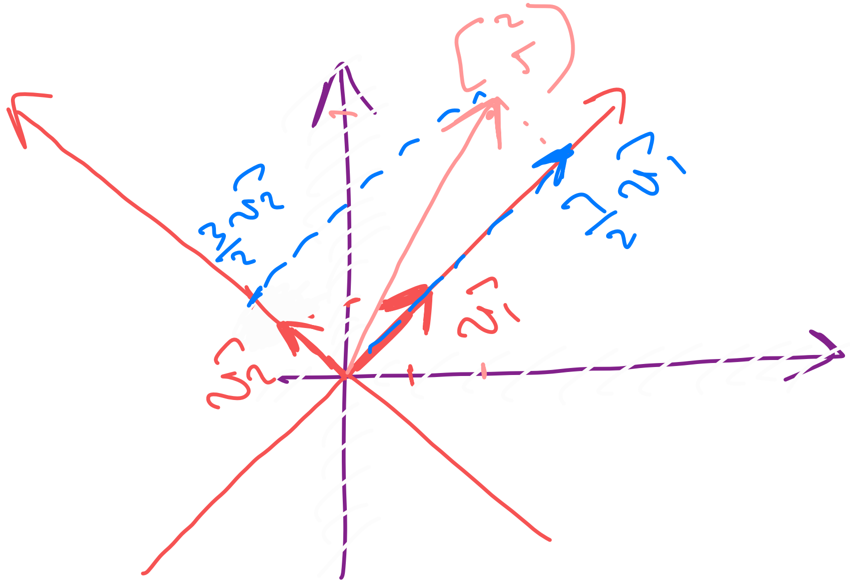
$\Rightarrow$

solve system

Intro

Consider the vector $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$





$$\begin{cases} 2 = \alpha - \beta \\ 5 = \alpha + \beta \end{cases} \xrightarrow[\text{solve}]{} \begin{array}{l} \alpha = 7/2 \\ \beta = 3/2 \end{array}$$

old basis: $\begin{bmatrix} 2 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Matrix? $\downarrow$

new basis: $\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \frac{7}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \frac{7}{2} \vec{v_1} + \frac{3}{2} \vec{v_2}$$

Change of Basis

Change of Basis

Let $\vec{x} \in \mathbb{R}^n$, and let $\beta_1 = \{\vec{v}_1, \ldots, \vec{v}_n\}$ and $\beta_2 = \{\vec{w}_1, \ldots, \vec{w}_n\}$ be bases for $\mathbb{R}^n$. What are the coordinates of $\vec{x}$?

unique $\longrightarrow$ $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$

$\neq$

$c_1, c_2, \ldots, c_n$ are the coordinates of $\vec{x}$ in β_1

unique $\longrightarrow$ $\vec{x} = d_1\vec{w}_1 + d_2\vec{w}_2 + \cdots + d_n\vec{w}_n$

$d_1, d_2, \ldots, d_n$ are the coordinates of $\vec{x}$ in β_2

ex:

$\begin{bmatrix} 2 \\ 5 \end{bmatrix} = 2\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{e}_1} + 5\underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{e}_2}$ uniquely when $\beta_1 = \{\vec{e}_1, \vec{e}_2\}$

$\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \frac{7}{2}\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{v}_1} + \frac{3}{2}\underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\vec{v}_2}$ uniquely when $\beta_2 = \{\vec{v}_1, \vec{v}_2\}$

Change of Basis

(representations are unique for a fixed basis, but $\vec{x}$ is the same)

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \quad = \quad \vec{x} = d_1\vec{w}_1 + d_2\vec{w}_2 + \cdots + d_n\vec{w}_n$$

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = d_1\vec{w}_1 + d_2\vec{w}_2 + \cdots + d_n\vec{w}_n$$

$$\underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}}_{\vec{c}} = \underbrace{\begin{bmatrix} \vec{w}_1 & \vec{w}_2 & \dots & \vec{w}_n \end{bmatrix}}_{W} \underbrace{\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}}_{\vec{d}}$$

V: matrix storing basis elements $\vec{v}_1 \dots \vec{v}_n$ (col form) "old basis"

$\vec{c}$: "old" coefficients of $\vec{x}$ in terms of $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$

W: matrix storing basis elements $(\vec{w}_1 \dots \vec{w}_n)$ "new basis"

$\vec{d}$: "new" coefficients of $\vec{x}$ in terms of $\mathcal{B}' = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n\}$

Change of Basis

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \qquad \vec{x} = d_1\vec{w}_1 + d_2\vec{w}_2 + \cdots + d_n\vec{w}_n$$

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = d_1\vec{w}_1 + d_2\vec{w}_2 + \cdots + d_n\vec{w}_n$$

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \vec{w}_1 & \vec{w}_2 & \dots & \vec{w}_n \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

$$V\vec{c} = W\vec{d}$$

old coeff. $\leftarrow$ $\vec{c} = V^{-1}W\vec{d} \rightarrow$ new coeff

new coeff $\leftarrow$ $\vec{d} = W^{-1}V\vec{c} \rightarrow$ old coeff

Example

Let $\beta_1 = \underbrace{\left\{ \overset{\vec{v}_1}{\begin{bmatrix} -9 \\ 1 \end{bmatrix}}, \overset{\vec{v}_2}{\begin{bmatrix} -5 \\ -1 \end{bmatrix}} \right\}}_{\text{spans } \mathbb{R}^2}$ and $\beta_2 = \underbrace{\left\{ \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \end{bmatrix} \right\}}_{\text{spans } \mathbb{R}^2}$

Goal: coeff of $\begin{bmatrix} -13 \\ 3 \end{bmatrix}$ in β_2?

Change of basis matrix from β_1 to β_2:

ex: $\begin{bmatrix} -13 \\ 3 \end{bmatrix} = 2\begin{bmatrix} -9 \\ 1 \end{bmatrix} - \begin{bmatrix} -5 \\ -1 \end{bmatrix} = 2\vec{v}_1 + (-1)\vec{v}_2$

so in β_1, coefficients of $\begin{bmatrix} -13 \\ 3 \end{bmatrix}$ are $2, -1$

$V\vec{c} = W\vec{d}$ ← new coeff

V: old basis matrix; $\vec{c}$: old coeff; W: new basis matrix

$\Rightarrow \vec{d} = W^{-1}V\vec{c}$

$$\begin{bmatrix} -9 & -5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

need these

$$\Rightarrow \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -4 & -5 \end{bmatrix}^{-1} \begin{bmatrix} -9 & -5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

(from β_1 to β_2)

Example

Let $\beta_1 = \left\{ \begin{bmatrix} -9 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ -1 \end{bmatrix} \right\}$ and $\beta_2 = \left\{ \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \end{bmatrix} \right\}$

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 3 \\ -4 & -5 \end{bmatrix}^{-1}}_{\text{inverse shortcut}} \begin{bmatrix} -9 & -5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -5/7 & -3/7 \\ 4/7 & 1/7 \end{bmatrix} \begin{bmatrix} -9 & -5 \\ 1 & -1 \end{bmatrix}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}}_{\substack{\text{change of basis} \\ \text{matrix from } \beta_1 \text{ to } \beta_2}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

What does this mean?

old: $\begin{bmatrix} -13 \\ 3 \end{bmatrix} = \underbrace{2}_{c_1} \begin{bmatrix} -9 \\ 1 \end{bmatrix} + \underbrace{(-1)}_{c_2} \begin{bmatrix} -5 \\ -1 \end{bmatrix}$

new: $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \end{bmatrix} \begin{matrix} \leftarrow d_1 \\ \leftarrow d_2 \end{matrix}$

check:

$$8 \begin{bmatrix} 1 \\ -4 \end{bmatrix} + (-7) \begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} -13 \\ 3 \end{bmatrix} \checkmark$$

Example

Let $\beta_1 = \left\{ \begin{bmatrix} -9 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ -1 \end{bmatrix} \right\}$ and $\beta_2 = \left\{ \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \end{bmatrix} \right\}$

Change of basis from β_2 to β_1

Repeat the work by treating vectors in β_2 as "old" and vectors in β_1 as "new".

we end up with matrix: $\begin{bmatrix} -3/2 & -2 \\ 5/2 & 3 \end{bmatrix}$

Note:

$$A = M_{\beta_1 \to \beta_2} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix} \xleftrightarrow{\text{inverses!}} M_{\beta_2 \to \beta_1} = \begin{bmatrix} -3/2 & -2 \\ 5/2 & 3 \end{bmatrix} = A^{-1}$$