NYU

Linear Algebra

Lecture 21

# Introduction to Linear Transformations

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# Strang Section 8.1 – The Idea of a Linear Transformation and Section 8.2 – The Matrix of a Linear Transformation

Course notes adapted from N. Hammoud's NYU lecture notes.

# Linear Transformations

# What is a Linear Transformation?

**Def.**: Let $V$ and $W$ be two vector spaces over a field $\mathbb{F}$ ($\mathbb{R}$ in our case). We call a function $T : V \to W$ a linear transformation if for all $\vec{v}, \vec{w} \in V$ and $c \in \mathbb{F}$ ($c \in \mathbb{R}$), we have:

(i) $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$

(ii) $T(c\vec{v}) = cT(\vec{v})$

can combine:

$T(c\vec{v} + d\vec{w}) = cT(\vec{v}) + dT(\vec{w})$

**Properties:**

- $T(\vec{0}) = \vec{0}$
- $T(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1T(\vec{v}_1) + \cdots + c_nT(\vec{v}_n)$

$c_1, \cdots + c_n \in \mathbb{F}$

$\vec{v}_1, \cdots + \vec{v}_n \in V$

# Examples

$$\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \boxed{T} \longrightarrow \begin{bmatrix} 2x+y \\ x \end{bmatrix}$$

$T : \mathbb{R}^2 \to \mathbb{R}^2 \qquad T \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2v_1 + v_2 \\ v_1 \end{bmatrix}$

ex: $T \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2(-1)+1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

Is $T$ a linear transformation?

Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathbb{R}^2$, and let $c \in \mathbb{R}$.

(i) $\underline{T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})}$

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow T[\vec{v}] = \begin{bmatrix} 2v_1 + v_2 \\ v_1 \end{bmatrix}$

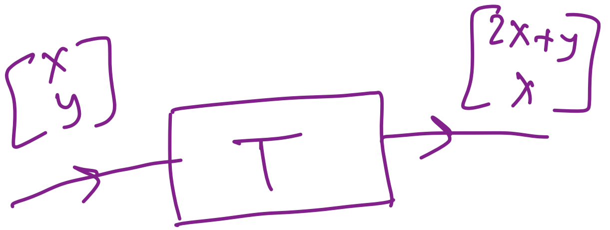
$\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \Rightarrow T[\vec{w}] = \begin{bmatrix} 2w_1 + w_2 \\ w_1 \end{bmatrix}$

$T[\vec{v}] + T[\vec{w}] = \begin{bmatrix} 2v_1 + v_2 + 2w_1 + w_2 \\ v_1 + w_1 \end{bmatrix}$

$\vec{v} + \vec{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix} \Rightarrow T[\vec{v} + \vec{w}] = \begin{bmatrix} 2(v_1 + w_1) + (v_2 + w_2) \\ (v_1 + w_1) \end{bmatrix} = \begin{bmatrix} 2v_1 + 2w_1 + v_2 + w_2 \\ v_1 + w_1 \end{bmatrix}$

Thus $T[\vec{v} + \vec{w}] = T[\vec{v}] + T[\vec{w}] = \begin{bmatrix} 2v_1 + v_2 + 2w_1 + w_2 \\ v_1 + w_1 \end{bmatrix}$

# Examples




$T : \mathbb{R}^2 \to \mathbb{R}^2 \qquad T \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2v_1 + v_2 \\ v_1 \end{bmatrix}$

ex: $T\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2(-1)+1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

Is $T$ a linear transformation?

Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathbb{R}^2$, and let $c \in \mathbb{R}$.

(ii) $\underline{T(c\vec{v}) = cT(\vec{v})}$

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow T[\vec{v}] = \begin{bmatrix} 2v_1 + v_2 \\ v_1 \end{bmatrix}$

$c\vec{v} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix} \Rightarrow T[c\vec{v}] = \begin{bmatrix} 2cv_1 + cv_2 \\ cv_1 \end{bmatrix} = c\begin{bmatrix} 2v_1 + v_2 \\ v_1 \end{bmatrix} = cT[\vec{v}]$ ✓

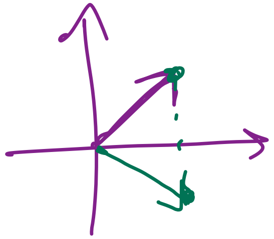
Yes, T is linear

# Examples

Note: this transformation represents a reflection about x-axis

$T : \mathbb{R}^2 \to \mathbb{R}^2 \qquad T\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix}$

ex: $T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$



Is $T$ a linear transformation?

Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathbb{R}^2$, and let $c \in \mathbb{R}$.

$$T[\underbrace{c\vec{v} + d\vec{w}}_{c\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + d\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}}] = T\begin{bmatrix} cv_1 + dw_1 \\ cv_2 + dw_2 \end{bmatrix}$$

$$= \begin{bmatrix} cv_1 + dw_1 \\ -cv_2 - dw_2 \end{bmatrix} = \begin{bmatrix} cv_1 \\ -cv_2 \end{bmatrix} + \begin{bmatrix} dw_1 \\ -dw_2 \end{bmatrix}$$

$$= c\begin{bmatrix} v_1 \\ -v_2 \end{bmatrix} + d\begin{bmatrix} w_1 \\ -w_2 \end{bmatrix}$$

$$= c\,T[\vec{v}] + d\,T[\vec{w}]$$

$\therefore$ T is linear

EFY: what if $T\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 + 3 \\ -v_2 \end{bmatrix}$?

# Examples

$$T : \mathbb{R}^2 \to \mathbb{R}^2 \qquad T\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 v_2 \\ v_1 \end{bmatrix}$$

hint that T is not linear.

Is $T$ a linear transformation?

Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathbb{R}^2$, and let $c \in \mathbb{R}$.

T is not linear:

$\vec{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \Rightarrow T[\vec{v}] = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

$\vec{w} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \Rightarrow T[\vec{w}] = \begin{bmatrix} 9 \\ 3 \end{bmatrix}$

$\Bigg\}\ T[\vec{v}] + T[\vec{w}] = \begin{bmatrix} 13 \\ 5 \end{bmatrix}$

$\vec{v} + \vec{w} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \Rightarrow T[\vec{v} + \vec{w}] = \begin{bmatrix} 25 \\ 5 \end{bmatrix} \neq \begin{bmatrix} 13 \\ 5 \end{bmatrix}$

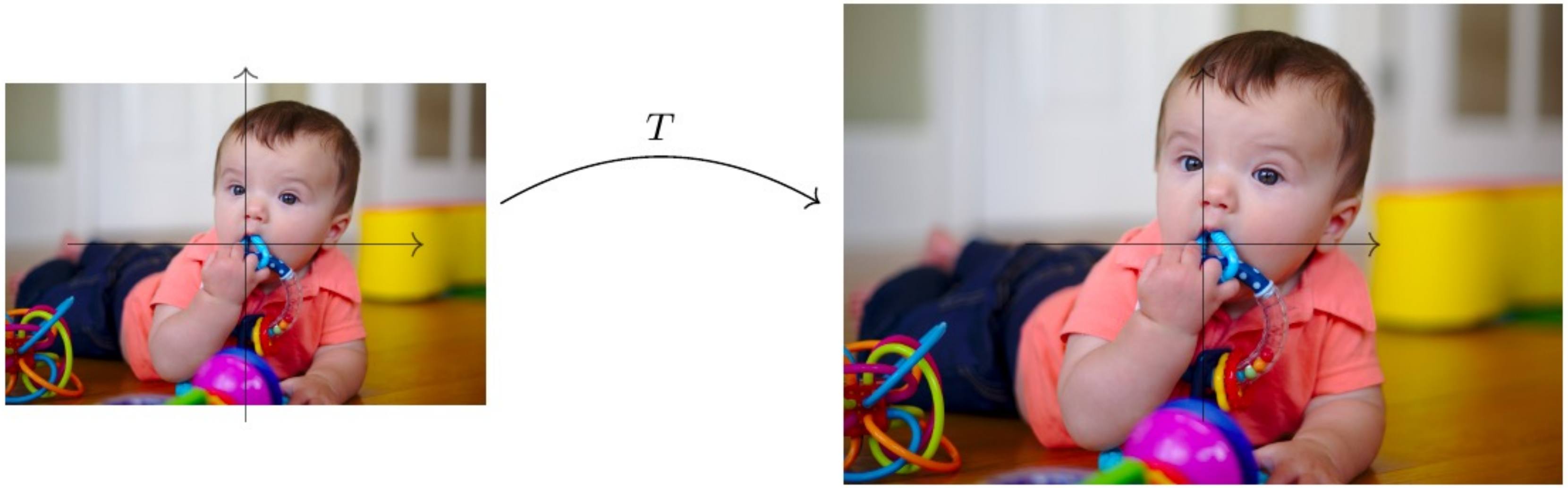
# Application

Define $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x) = 1.5x$. Is $T$ linear? Check:

$$T(u+v) = 1.5(u+v) = 1.5u + 1.5v = T(u) + T(v)$$
$$T(cv) = 1.5(cv) = c(1.5v) = c(Tv).$$

So $T$ satisfies the two equations, hence $T$ is linear.

This is called **dilation** or **scaling** (by a factor of 1.5). Picture:



# Space of Polynomials of degree $n$

$\mathbb{P}_n$ : space of polynomials of degree $n$. (These are vector spaces)

<u>ex:</u> $\mathbb{P}_2$ contains all polynomials of degree 2:

$B_{\mathbb{P}_2} = \{1, x, x^2\}$

$f(x) = \frac{3}{2}x^2 + 4x - \underbrace{7}_{7x^0 = 7(1)}$ $\qquad$ $g(x) = \underbrace{6x - 1}_{0x^2 + 6x - 1(1)}$

$f(x) = \begin{pmatrix} -7 & 4 & 3/2 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix}$ $\qquad$ $g(x) = \begin{pmatrix} -1 & 6 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix}$

$B_{\mathbb{P}_5} = \{1, x, x^2, x^3, x^4, x^5\}$

$\mathbb{P}_5$ contains all polynomials of degree 5:

$h(x) = x^5 - 4x^3 + 6x^2 - 3 = \begin{pmatrix} -3 & 0 & 6 & -4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{pmatrix}$

# Example

Define $T : \mathbb{P}_n \to \mathbb{P}_{n-1}$ such that $T[p(x)] = 5p'(x)$. Is $T$ linear?

ex: $T(x^2 - 6x) = 5(2x - 6) = 10x - 30$

let $p(x)$ and $q(x)$ be polynomials of degree $n$ $(p(x), q(x) \in \mathbb{P}_n)$

(i) $T[p(x)] = 5p'(x)$

$T[q(x)] = 5q'(x)$

$\Big\}$ $T[p(x)] + T[q(x)] = 5p'(x) + 5q'(x)$

derivative operator is linear

(scaler product)

$T[p(x) + q(x)] = 5(p(x) + q(x))' = 5(p'(x) + q'(x)) = 5p'(x) + 5q'(x)$

(ii) let $c \in \mathbb{R}$. Then $T[cp(x)] = 5[cp(x)]' = 5[cp'(x)]$

$= c[5p'(x)] = cT[p(x)]$

# Matrix of a Linear Transformation

# The Matrix of a Linear Transformation

Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ can be written as $T(\vec{v}) = A\vec{v}$ for a unique $m \times n$ matrix $A$ and any $\vec{v} \in \mathbb{R}^n$.

The transformation $T(\vec{v}) = A\vec{v}$ is indeed linear since: $(\vec{v}, \vec{\omega} \in \mathbb{R}^n, c, d \in \mathbb{R})$

$$T(c\vec{v} + d\vec{\omega}) = A(c\vec{v} + d\vec{\omega}) = Ac\vec{v} + Ad\vec{\omega} = c(A\vec{v}) + d(A\vec{\omega}) = cT(\vec{v}) + dT(\vec{\omega})$$

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So any matrix represents some linear transformation.

ex: $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ 3 & 1 \end{pmatrix}$ $A$ is $3 \times 2$, so $A$ represents some transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$

$(A\vec{x} = \vec{b}$ so $T(\vec{x}) = A\vec{x} = \vec{b})$

if $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; $A\vec{x} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ -x_2 \\ 3x_1 + x_2 \end{pmatrix}$

so $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ -x_2 \\ 3x_1 + x_2 \end{bmatrix}$

ex: $T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

# Examples

$$T : \mathbb{R}^2 \to \mathbb{R}^2 \qquad T\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix}$$

We already showed that T is linear (a few slides ago). We can find the matrix A associated with T.

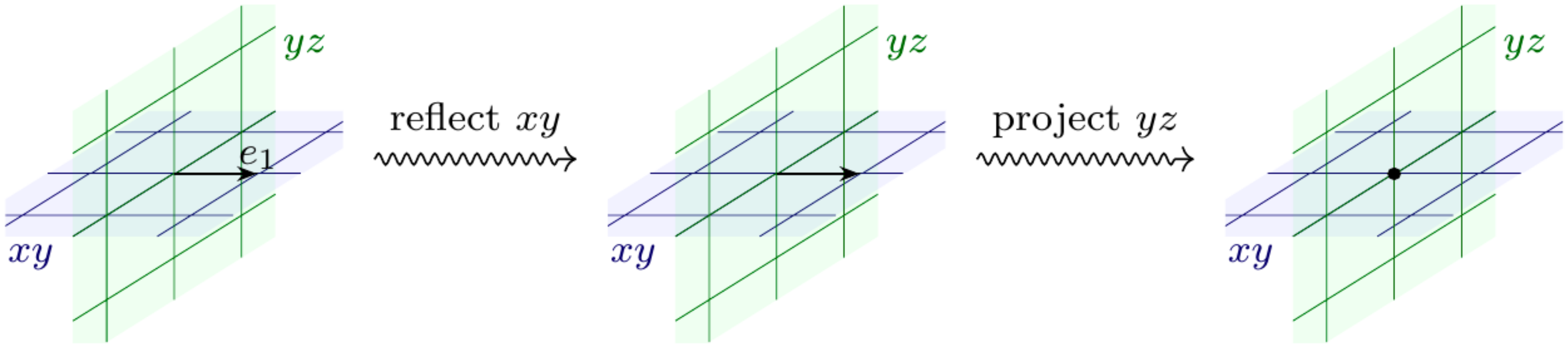
We need 2 vectors in $\mathbb{R}^2$, as long as they are a basis (linearly indep)

simplest case: $\beta_{\mathbb{R}^2} = \{\vec{e}_1, \vec{e}_2\}$

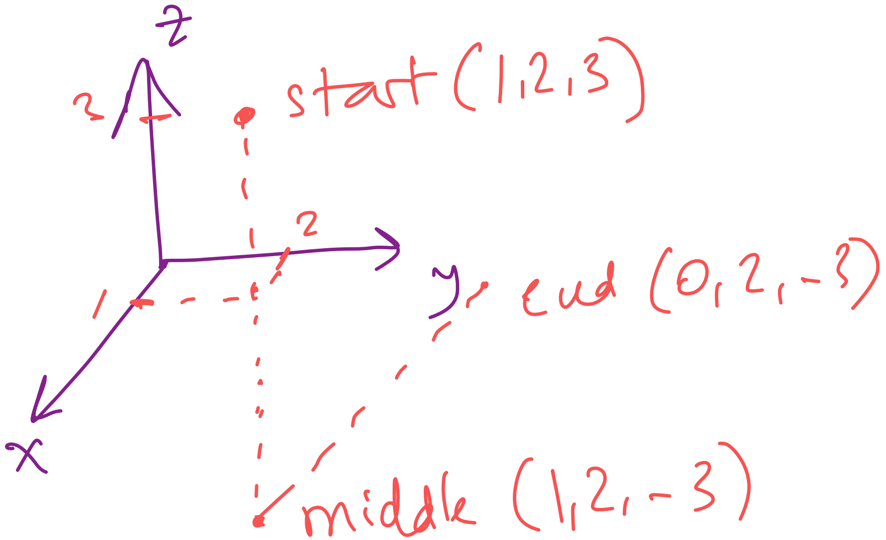
$$\left.\begin{aligned} T[\vec{e}_1] &= T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ T[\vec{e}_2] &= T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \end{aligned}\right\} \quad A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \to \text{standard matrix of } T$$

# Examples

What is the matrix for the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ that reflects through the $xy$-plane and then projects onto the $yz$-plane?




$$e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \Rightarrow \quad T(e_1) = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$



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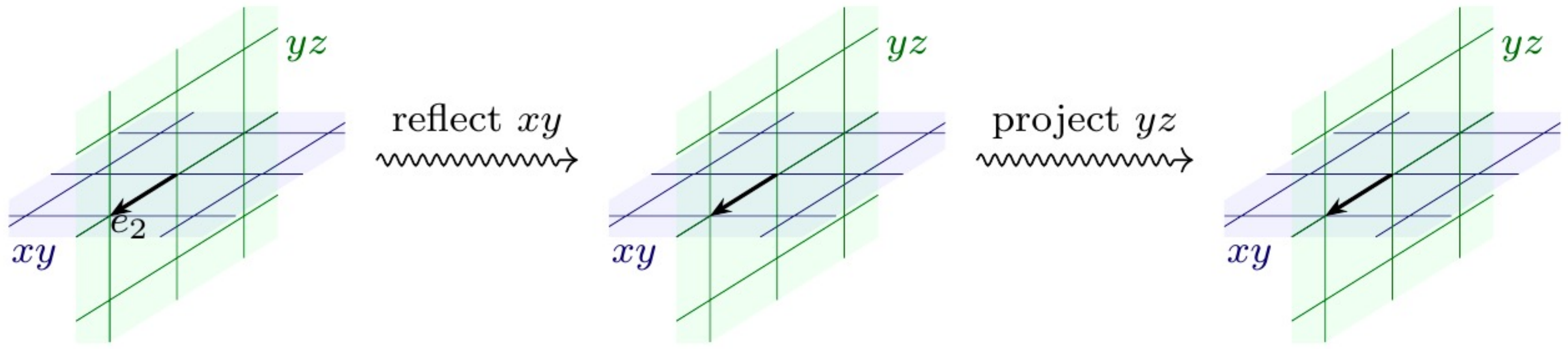
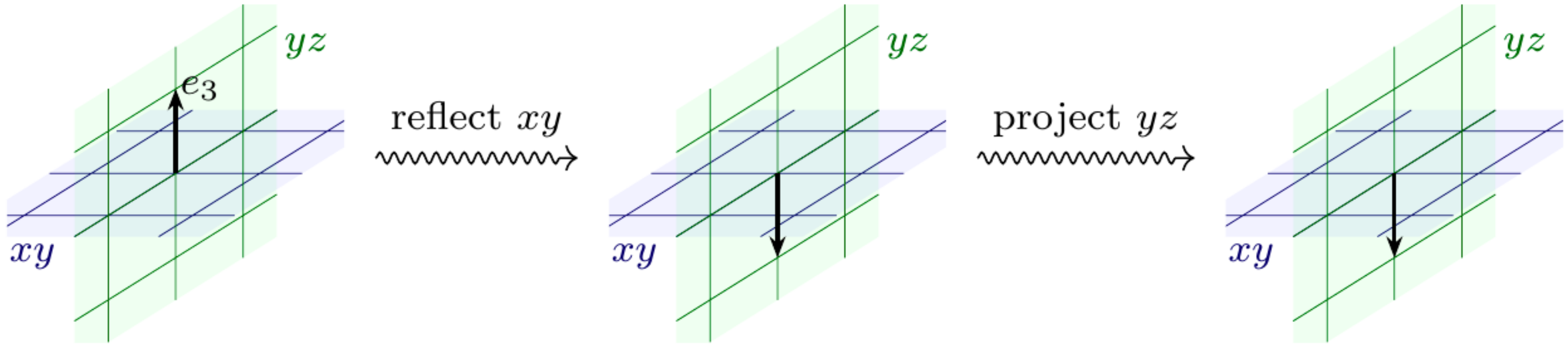



$e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow$

$$T(e_2) = e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

# Examples

What is the matrix for the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ that reflects through the $xy$-plane and then projects onto the $yz$-plane?




$$e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$T(e_3) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

# Examples

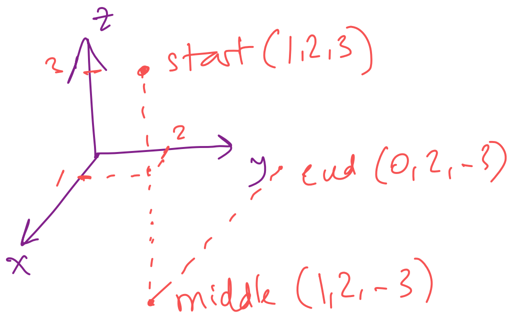
What is the matrix for the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ that reflects through the $xy$-plane and then projects onto the $yz$-plane?

$$\left.\begin{aligned} T(e_1) &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ T(e_2) &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ T(e_1) &= \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \end{aligned}\right\} \implies A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

matrix using standard bases for $\mathbb{R}^3$ and $\mathbb{R}^3$

i.e. standard matrix

check: $T\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = A\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$



# Examples

$T : \mathbb{R}^4 \to \mathbb{R}^3$ Construct $A$ such that

$$T\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}2\\3\\1\end{bmatrix}$$

$$T\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}-3\\-2\\-2\end{bmatrix}$$

$$T\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}-2\\-3\\-1\end{bmatrix}$$

$$T\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\5\\-1\end{bmatrix}$$

# Examples

$T : \mathbb{R}^4 \to \mathbb{R}^3$ Construct $A$ such that

$$T\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}2\\3\\1\end{bmatrix} \to T(\vec{e}_1) = \begin{bmatrix}2\\3\\1\end{bmatrix}$$

$$T\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}-3\\-2\\-2\end{bmatrix} \to T(\vec{e}_3) = \begin{bmatrix}-3\\-2\\-2\end{bmatrix}$$

$$T\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}-2\\-3\\-1\end{bmatrix} \to T(\vec{e}_2) = \begin{bmatrix}-2\\-3\\-1\end{bmatrix}$$

$$T\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\5\\-1\end{bmatrix} \to T(\vec{e}_4) = \begin{bmatrix}0\\5\\-1\end{bmatrix}$$

# Examples

$$T\begin{bmatrix}1\\0\\0\\0\end{bmatrix}=\begin{bmatrix}2\\3\\1\end{bmatrix}\rightarrow T(\vec{e}_1)=\begin{bmatrix}2\\3\\1\end{bmatrix}$$

$$T\begin{bmatrix}0\\0\\1\\0\end{bmatrix}=\begin{bmatrix}-3\\-2\\-2\end{bmatrix}\rightarrow T(\vec{e}_3)=\begin{bmatrix}-3\\-2\\-2\end{bmatrix}$$

$$T\begin{bmatrix}0\\1\\0\\0\end{bmatrix}=\begin{bmatrix}-2\\-3\\-1\end{bmatrix}\rightarrow T(\vec{e}_2)=\begin{bmatrix}-2\\-3\\-1\end{bmatrix}$$

$$T\begin{bmatrix}0\\0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\5\\-1\end{bmatrix}\rightarrow T(\vec{e}_4)=\begin{bmatrix}0\\5\\-1\end{bmatrix}$$

$$\text{Let } \vec{v}=\begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix}\in\mathbb{R}^4 \implies \vec{v}=v_1\begin{bmatrix}1\\0\\0\\0\end{bmatrix}+v_2\begin{bmatrix}0\\1\\0\\0\end{bmatrix}+v_3\begin{bmatrix}0\\0\\1\\0\end{bmatrix}+v_4\begin{bmatrix}0\\0\\0\\1\end{bmatrix}=v_1\vec{e}_1+v_2\vec{e}_2+v_3\vec{e}_3+v_4\vec{e}_4$$

(by the fact that a basis for $\mathbb{R}^4$ is $\{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\}$

Next, apply $T$ on both sides of the equation

# Examples

$$T\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}2\\3\\1\end{bmatrix} \rightarrow T(\vec{e}_1) = \begin{bmatrix}2\\3\\1\end{bmatrix}$$

$$T\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}-3\\-2\\-2\end{bmatrix} \rightarrow T(\vec{e}_3) = \begin{bmatrix}-3\\-2\\-2\end{bmatrix}$$

$$T\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}-2\\-3\\-1\end{bmatrix} \rightarrow T(\vec{e}_2) = \begin{bmatrix}-2\\-3\\-1\end{bmatrix}$$

$$T\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\5\\-1\end{bmatrix} \rightarrow T(\vec{e}_4) = \begin{bmatrix}0\\5\\-1\end{bmatrix}$$

$$T(\vec{v}) = T\begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix} = T(v_1\vec{e}_1 + v_2\vec{e}_2 + v_3\vec{e}_3 + v_4\vec{e}_4) = \underbrace{v_1T(\vec{e}_1) + v_2T(\vec{e}_2) + v_3T(\vec{e}_3) + v_4T(\vec{e}_4)}$$

$= T(v_1\vec{e}_1 + v_2\vec{e}_2 + v_3\vec{e}_3 + v_4\vec{e}_4)$

since $T$ is linear

# Examples

$$T\begin{bmatrix}1\\0\\0\\0\end{bmatrix}=\begin{bmatrix}2\\3\\1\end{bmatrix}\rightarrow T(\vec{e}_1)=\begin{bmatrix}2\\3\\1\end{bmatrix}$$

$$T\begin{bmatrix}0\\0\\1\\0\end{bmatrix}=\begin{bmatrix}-3\\-2\\-2\end{bmatrix}\rightarrow T(\vec{e}_3)=\begin{bmatrix}-3\\-2\\-2\end{bmatrix}$$

$$T\begin{bmatrix}0\\1\\0\\0\end{bmatrix}=\begin{bmatrix}-2\\-3\\-1\end{bmatrix}\rightarrow T(\vec{e}_2)=\begin{bmatrix}-2\\-3\\-1\end{bmatrix}$$

$$T\begin{bmatrix}0\\0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\5\\-1\end{bmatrix}\rightarrow T(\vec{e}_4)=\begin{bmatrix}0\\5\\-1\end{bmatrix}$$

$$T(\vec{v})=T\begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix}=T(v_1\vec{e}_1+v_2\vec{e}_2+v_3\vec{e}_3+v_4\vec{e}_4)=v_1\begin{bmatrix}2\\3\\1\end{bmatrix}+v_2\begin{bmatrix}-2\\-3\\-1\end{bmatrix}+v_3\begin{bmatrix}-3\\-2\\-2\end{bmatrix}+v_4\begin{bmatrix}0\\5\\-1\end{bmatrix}$$

# Examples

$T: \mathbb{R}^4 \to \mathbb{R}^3$

$$T\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}2\\3\\1\end{bmatrix} \to T(\vec{e}_1) = \begin{bmatrix}2\\3\\1\end{bmatrix}$$

$$T\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}-3\\-2\\-2\end{bmatrix} \to T(\vec{e}_3) = \begin{bmatrix}-3\\-2\\-2\end{bmatrix}$$

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$$T\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\5\\-1\end{bmatrix} \to T(\vec{e}_4) = \begin{bmatrix}0\\5\\-1\end{bmatrix}$$

$$T(\vec{v}) = T\begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix} = T(v_1\vec{e}_1 + v_2\vec{e}_2 + v_3\vec{e}_3 + v_4\vec{e}_4) = \underbrace{\begin{bmatrix}2 & -2 & -3 & 0\\3 & -3 & -2 & 5\\1 & -1 & -2 & -1\end{bmatrix}}_{3\times 4}\underbrace{\begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix}}_{4\times 1} = \underbrace{\begin{bmatrix}2v_1 - 2v_2 - 3v_3\\3v_1 - 3v_2 - 2v_3 + 5v_4\\v_1 - v_2 - 2v_3 - v_4\end{bmatrix}}_{3\times 1}$$

$\Rightarrow$ $T: \mathbb{R}^4 \to \mathbb{R}^3$ such that $T\begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix} = \begin{bmatrix}2v_1 - 2v_2 - 3v_3\\3v_1 - 3v_2 - 2v_3 + 5v_4\\v_1 - v_2 - 2v_3 - v_4\end{bmatrix} \to \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

# How to Compute the Matrix $A$?

In general, if $T$ is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, then the $m \times n$ matrix $A$ corresponding to $T$ is given by

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \dots & T(\vec{e}_n) \end{bmatrix}$$

This is true if both the input and output bases are standard bases, i.e., $\beta_{in} = \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ and $\beta_{out} = \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_m\}$.

If $\beta_{in}$ (basis for the domain of $T$) or $\beta_{out}$ (basis for the codomain of $T$) is not standard, then this does not apply (Lecture 22).

# Derivative Example

$T: \mathbb{P}_2 \to \mathbb{P}_1$
$p(x) \mapsto p'(x)$

In words, T transforms a polynomial of degree 2 (in span $\{1, x, x^2\}$) to its derivative, a polynom. of degree 1 (in span $\{1, x\}$)

Input basis : $\mathcal{B}_{in} = \underbrace{\{1, x, x^2\}}_{\text{"}e_1, e_2, e_3\text{"}}$

Output basis : $\mathcal{B}_{out} = \{1, x\}$
"$e_1, e_2$"

$$T(1) = 0 = 0(1) + 0(x)$$
$$T(x) = \boxed{1} = \boxed{1}(1) + 0(x)$$
$$T(x^2) = 2x = 0(1) + 2(x)$$

Matrix : $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

ex: $T(\underbrace{3x^2 - 2x + 5}) = 6x - 2$ ; $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$

Input $\cancel{\begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}}$ $\begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$

# Integral Example

Optional

$$T: \mathbb{P}_2 \to \mathbb{P}_3$$

$$p(x) \mapsto \int_0^x p(t)\,dt$$

input basis $\{1, x, x^2\}$

output basis $\{1, x, x^2, x^3\}$

$$T(1) = \int_0^x 1\,dt = x = 0 \cdot 1 + 1 \cdot x + 0 \cdot x^2 + 0\,x^3$$

$$T(x) = \int_0^x t\,dt = \frac{x^2}{2} = 0 \cdot 1 + 0 \cdot x + \frac{1}{2} x^2 + 0\,x^3$$

$$T(x^2) = \int_0^x t^2\,dt = \frac{x^3}{3} = 0 \cdot 1 + 0 \cdot x + 0\,x^2 + \frac{1}{3} x^3$$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

<u>example</u>

$$\int_0^x \underbrace{(t - 3t^2)}_{\downarrow}\,dt$$

coefficient $\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$

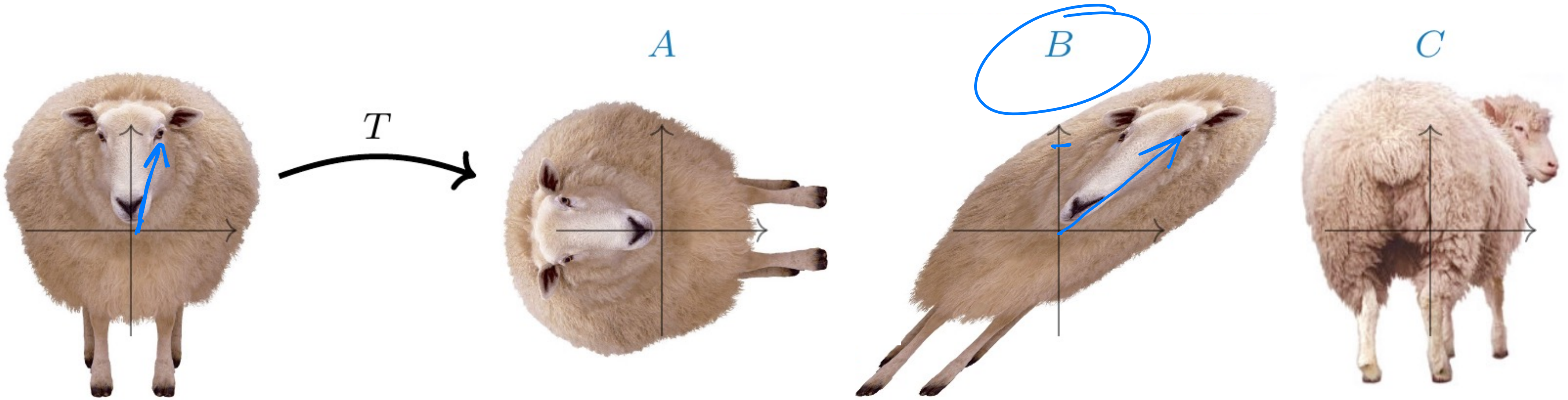
so $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ -1 \end{bmatrix}$

$$\int_0^x (t^2 - 3t^2)\,dt = \frac{x^2}{2} - \frac{3x^3}{3} = \frac{1}{2} x^2 - x^3$$

# Choose the Right Transformed Image

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $T(x) = Ax$, so $T\colon \mathbb{R}^2 \to \mathbb{R}^2$.

(This is a shear)




Find $\underbrace{T\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\text{or any vector that makes sense}} = A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

or, in general $T\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 + v_2 \\ v_2 \end{pmatrix}$