NYU

Linear Algebra

Lecture 20

Singular Value Decomposition

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Strang Chapter 7 – The Singular Value Decomposition

Course notes adapted from N. Hammoud's NYU lecture notes.

SVD

SVD

$$A = U\Sigma V^T$$

orthogonal matrix $U^{-1} = U^T$

"diagonal" matrix

orthogonal matrix $V^{-1} = V^T$

$$A^T A = (U\Sigma V^T)^T (U\Sigma V^T)$$

$$= V\Sigma^T U^T U \Sigma V^T$$

$$= V\Sigma^T \Sigma V^T$$

eigenvectors of $A^T A$

eigenvalues of $A^T A$

$$AA^T = (U\Sigma V^T)(U\Sigma V^T)^T$$

$$= U\Sigma V^T V \Sigma^T U^T$$

$$= U\Sigma\Sigma^T U^T$$

eigenvectors of AA^T

eigenvalues of AA^T

The Eigenvalues of A^TA and AA^T

$$\Sigma^T\Sigma = \begin{bmatrix} \sigma_1^2 & & & \\ & \ddots & & \\ & & \sigma_r^2 & \\ & & & 0 \end{bmatrix} \qquad \Sigma\Sigma^T = \begin{bmatrix} \sigma_1^2 & & & \\ & \ddots & & \\ & & \sigma_r^2 & \\ & & & 0 \end{bmatrix}$$

Therefore, the matrices A^TA and AA^T have the same nonzero eigenvalues.

$\sigma_1 = \sqrt{\lambda_1}$, where λ_1 is the largest eigenvalue of A^TA (or AA^T)

$\vdots$

$\sigma_r = \sqrt{\lambda_r}$, where λ_r is the smallest (nonzero) eigenvalue of A^TA (or AA^T)

Example

The Connection between *U* and *V*

$$A = U\Sigma V^T \implies AV = U\Sigma$$

$$A\vec{v}_1 = \sigma_1\vec{u}_1 \implies \vec{u}_1 = \frac{1}{\sigma_1}A\vec{v}_1$$

$$A\vec{v}_2 = \sigma_1\vec{u}_2 \implies \vec{u}_2 = \frac{1}{\sigma_2}A\vec{v}_2$$

$$\vdots$$

$$A\vec{v}_r = \sigma_1\vec{u}_r \implies \vec{u}_r = \frac{1}{\sigma_r}A\vec{v}_r$$

The remaining $\vec{u}_{r+1}, \ldots, \vec{u}_m$ are determined as eigenvectors of AA^T corresponding to zero eigenvalue.

How to find the SVD Decomposition?

1. Σ is the same size as A and has the singular values σ on its diagonal. The singular values are the square roots of the nonzero eigenvalues of A^TA or AA^T.

2. V contains the eigenvectors of A^TA.

3. U contains the eigenvectors of AA^T. But we will find the vectors $\vec{u}$ corresponding to nonzero eigenvalues using the SVD.

Example

Bases of the Four Fundamental Subspaces using the SVD

The Matrix V

$$A^T A = V\Sigma^T \Sigma V^T$$

$$A^T A\vec{v}_1 = \sigma_1^2 \vec{v}_1$$

$$A^T A\vec{v}_2 = \sigma_2^2 \vec{v}_2$$

$$\vdots$$

$$A^T A\vec{v}_r = \sigma_r^2 \vec{v}_r$$

$\{\vec{v}_1, \ldots, \vec{v}_r\} \perp \{\vec{v}_{r+1}, \ldots, \vec{v}_n\}$

$\{\vec{v}_1, \ldots, \vec{v}_r\}$ orthonormal basis for Row$A$

$$A^T A\vec{v}_{r+1} = \sigma_{r+1}^2 \vec{v}_{r+1} = \vec{0}$$

$$\vdots$$

$$A^T A\vec{v}_n = \sigma_n^2 \vec{v}_n = \vec{0}$$

$\vec{v}_{r+1}, \ldots, \vec{v}_n$ span $\text{Nul}(A^T A) = \text{Nul}A$

$\{\vec{v}_{r+1}, \ldots, \vec{v}_n\}$ orthonormal basis for Nul$A$

The Matrix U

$$AA^T = U\Sigma\Sigma^T U^T$$

$$AA^T\vec{u}_1 = \sigma_1^2\vec{u}_1$$

$$AA^T\vec{u}_2 = \sigma_2^2\vec{u}_2$$

$$\vdots$$

$$AA^T\vec{u}_r = \sigma_r^2\vec{u}_r$$

$\{\vec{u}_1, \ldots, \vec{u}_r\} \perp \{\vec{u}_{r+1}, \ldots, \vec{u}_m\}$

$\{\vec{u}_1, \ldots, \vec{u}_r\}$ orthonormal basis for Col$A$

$$AA^T\vec{u}_{r+1} = \sigma_{r+1}^2\vec{u}_{r+1} = \vec{0}$$

$$\vdots$$

$$AA^T\vec{u}_m = \sigma_m^2\vec{u}_m = \vec{0}$$

$\vec{u}_{r+1}, \ldots, \vec{u}_m$ span $\text{Nul}(AA^T) = \text{Nul}A^T$

$\{\vec{u}_{r+1}, \ldots, \vec{u}_m\}$ orthonormal basis for NulA^T

Summary and Examples

$\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r$	is an orthonormal basis for the **column space**
$\boldsymbol{u}_{r+1}, \ldots, \boldsymbol{u}_m$	is an orthonormal basis for the **left nullspace** $\boldsymbol{N}(A^{\mathrm{T}})$
$\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$	is an orthonormal basis for the **row space**
$\boldsymbol{v}_{r+1}, \ldots, \boldsymbol{v}_n$	is an orthonormal basis for the **nullspace** $\boldsymbol{N}(A)$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\frac{\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}}{\sqrt{10}} \begin{bmatrix} \sqrt{50} & 0 \\ 0 & 0 \end{bmatrix} \frac{\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}}{\sqrt{5}}$$