NYU

Linear Algebra

Lecture 19

Symmetric and Positive Definite Matrices

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Recap

$$A\overset{\text{eigen vector}}{\boxed{x}} = \overset{\text{eigenvalue}}{\boxed{\lambda}}x \qquad x \in \mathbb{R}^n. \qquad A \in \mathbb{R}^{n\times n} \qquad \lambda \in \mathbb{R}$$

- λ is the solution to $\overset{\text{poly}}{p(\lambda)} = \det(\lambda I - A) = 0$. λ may be complex number!
  x is in the $\mathrm{Nul}(A - \lambda I)$

- If we have n distinict eigen value $\lambda_1 \cdots \lambda_n$
  1. eigenvectors $x_1 \cdots x_n$ are linear independent $X = [x_1 \cdots x_n]$ is invertible
  2. $A = \underset{I}{X}\Lambda X^{-1}$ $\Lambda = \mathrm{diag}(\lambda_1 \cdots \lambda_n)$
     $X = [x_1 \cdots x_n]$ matrix of all eigenvectors

1. distinct eigenvalues mean . $\dim(\mathrm{Nul}(A-\lambda I))$ is always 1.
   ↳ the equation $Ax = \lambda x$ only have a "single" solution "single" ⟺ same direction

Example. $A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}$ (in eigen slide)

$$\mathrm{Nul}(A - 2I) = \mathrm{span}\left\{ \overset{\vec{v}_1}{\begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}}, \overset{\vec{v}_2}{\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}} \right\}$$

$\vec{v}_1$ and $\vec{v}_2$ both have eigenvalue 2 it's not the case of distinct eigenvalue

But this still can be diagonalized!

Example $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $p(\lambda) = \det\left(\begin{pmatrix} 1-\lambda & 1 \\ & 1-\lambda \end{pmatrix}\right) = (1-\lambda)^2$. $\underbrace{\lambda_1 = 1 . \lambda_2 = 1}_{\text{Two same eigenvalue}}$

but $B - \lambda I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. $\mathrm{Nul}(B - I) = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$

This is a 2×2 . only have 1 eigenvectors! so we can't do digonalization!

(Not Required!) All Matrix is not diagonalizable will similar to B (Jordan Form)

Similar Matrix A and B are similar means

$$A = X B X^{-1} \qquad (X \text{ is invertible})$$

1. A and B have the same eigenvalue.

$$Ax = \lambda x \Rightarrow XBX^{-1}x = \lambda x \Rightarrow B(X^{-1}x) = \lambda(X^{-1}x)$$

if x is A's eigenvector, the $X^{-1}x$ is B's eigenvector!

Recap

$$\overset{\mathbb{R}^n}{A}\vec{x} = \lambda \vec{x} \qquad (\overset{\text{eigenvalue}}{\lambda}, \overset{\text{eigenvector}}{\vec{x}})$$

- λ is the solution to $\det(A-\lambda I) = 0$ $\longrightarrow$ $x \in Nul(A-\lambda I)$

n-th order polynomial | "$\dim(Nul(A-\lambda I))$"

How many eigenvectors we have for eigenvalue λ.

might be complex number $\leftarrow$ n solutions

n×n matrix will always have n eigenvalues.

- If the n eigenvalues are different. $\vec{x}_1 \cdots \vec{x}_n$ n-eigenvectors will become linear independent. $X = [\vec{x}_1, \cdots \vec{x}_n]$ is invertible

$$A = X \boxed{\Lambda} X^{-1}. \quad \Lambda = \operatorname{diag}\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

Example 1) $A = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$ $p(\lambda) = \det\left(\begin{pmatrix} 1-\lambda & \\ & 1-\lambda \end{pmatrix}\right) = (1-\lambda)^2 \Rightarrow \lambda_1 = 1.\ \lambda_2 = 1$

2×2 matrix have 2 eigen-value.

2) $B = \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}$ $p(\lambda) = \det\left(\begin{pmatrix} 1-\lambda & 1 \\ & 1-\lambda \end{pmatrix}\right) = (1-\lambda)^2 \cdot \Rightarrow \lambda_1 = 1.\ \lambda_2 = 1$

Eigenvector of Matrix A = $Nul(A-I) = Nul\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)$ $\leftarrow$ dim is 2 / 2 eigenvector

Eigenvector of Matrix B = $Nul(B-I) = Nul\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$ $\leftarrow$ dim is 1 / 1 eigenvector! $\quad span\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$

A - have 2-eigenvectors. so we can form the matrix X $\Rightarrow$ A is diagonalizable.

B - have 1 - eigenvector ... can't ... X $\Rightarrow$ B is NOT diagonalizable.

(Not Required) all matrix can't be diagonalized will similar to matrix B (Jordan Form)

Similar Matrix

$A = X B X^{-1}$. A and B are similar!

- They have the same eigen value. but they don't have the same eigenvector!

2. If we know $A = XBX^{-1}$, How to find X

A's eigenvector are $[a_1, a_2, \cdots a_n]$

B's eigenvector are $[b_1, b_2 \cdots b_n]$

$$= [X^{-1}a_1 \cdots X^{-1}a_n]$$

$$= X^{-1}[a_1 \cdots a_n]$$

$$\Rightarrow X^{-1}[a_1, a_2, \cdots a_n] = [b_1 \cdots b_n]$$

$$\Rightarrow X = [a_1 \cdots a_n][b_1 \cdots b_n]^{-1}$$

3. $Tr(A) = Tr(B) = \lambda_1 + \cdots + \lambda_n$

$det(A) = det(B) = \lambda_1 \cdots \lambda_n$

Symmetric Matrix $A = A^T$.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

① Symmetric Matrix will always have real eigenvalue, and it can be diagonalized

Motivation Qudratic function. $f(x_1, x_2) = a_1 x_1^2 + a_2 x_1^2 + a_3 x_1 x_2$

$A \in \mathbb{R}^{2\times 2} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$

let's calculate. $x^T A x = \underset{1\times 2}{(x_1, x_2)} \underset{2\times 2}{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}} \underset{2\times 1}{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} \rightarrow$ 1×1 matrix (real number)

$$= (x_1, x_2) \begin{pmatrix} a_{11} x_1 + a_{12} x_2 \\ a_{21} x_1 + a_{22} x_2 \end{pmatrix} = x_1 (a_{11} x_1 + a_{12} x_2) + x_2 (a_{21} x_1 + a_{22} x_2)$$

let $a_{12} = a_{21} = \frac{1}{2} a_3$!!! $\Leftrightarrow$ A is symmetric

$$= a_{11} x_1^2 + (a_{12} + a_{21}) x_1 x_2 + a_{22} x_1^2$$

Symmetric Matrix $\leftrightarrow$ Qudratic function.

Example $f(x_1, x_2) = x_1^2 + 2 x_1 x_2 + x_2^2 = (x_1 + x_2)^2$.

$$x_1^2 + 2 x_1 x_2 + x_2^2 = (x_1 \;\; x_2) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\lambda_1 = 2$ $x_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\lambda_2 = 0$ $x_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(normalize to unit vector!) (normalize to unit vector!)

$X = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$ X is an orthogonal Matrix !! ← This is always True for Symmetrix.

X is orthognal $X^{-1} = X^T$.

$$A = X \Lambda X^{-1} = X \Lambda X^T$$

$$= [x_1 \;\; x_2] \begin{bmatrix} 2 & \\ & 0 \end{bmatrix} \begin{bmatrix} x_1^T \\ x_2^T \end{bmatrix} = [2x_1 \;\; 0x_2] \begin{bmatrix} x_1^T \\ x_2^T \end{bmatrix}$$

$$= \underset{2\times 1}{2x_1} \underset{1\times 2}{x_1^T} + \underset{2\times 1}{0 x_2} \underset{1\times 2}{x_2^T} \rightarrow 2\times 2.$$

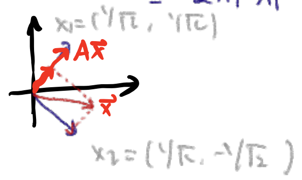
$\det(A - \lambda I)$

$= \det \begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix}$

$= (1-\lambda)(1-\lambda) - 1\times 1$

$= \lambda^2 - 2\lambda$

$$A = 2 \underbrace{x_1 x_1^T}_{\text{project matrix project to } x_1} + 0 \underbrace{x_2 x_2^T}_{\text{project matrix project to } x_2}.$$



Qudratic Function

$$f(x_1, x_2) = x_1^2 + 2x_1 x_2 + x_2^2 = x^T A x = x^T (2 x_1 x_1^T + 0 x_2 x_2^T) x$$

$$= 2 \underbrace{x^T x_1 \, x_1^T x}_{\text{real numbers } (x \cdot x_1)} + 0 \cdot \underbrace{x^T x_2 \, x_2^T x}_{\text{real number } (x \cdot x_2)} = 2 (x \cdot \underset{x_1 = (1/\sqrt{2}, 1/\sqrt{2})}{x_1})^2 + (x \cdot \underset{(1/\sqrt{2}, -1/\sqrt{2})}{x_2})^2$$

$$= 2 \left(1/\sqrt{2}\, x_1 + 1/\sqrt{2}\, x_2 \right)^2 + 0 \left(1/\sqrt{2}\, x_1 - 1/\sqrt{2}\, x_2 \right)^2$$

There are all square functions !

$$\uparrow \qquad \qquad \uparrow$$

$$(x_1 + x_2)^2 + 0 \cdot (x_1 - x_2)^2$$

just true for vectors $x_1^T x_2 = x_1^T x_1$

$$x_1^T (\lambda x_2)$$

$$= (x_{11} \;\; x_{12}) \begin{pmatrix} \lambda x_{21} \\ \lambda x_{22} \end{pmatrix}$$

$$= \lambda x_{11} x_{21} + \lambda x_{12} x_{22}$$

Symmetric Matrix. $A = A^T$.

① if A is symmetric, Then A can always be diagonalized
(provide proof on website)
and A's eigenvalues are always real numbers!

Motivation!

Qudratic function. $f(x) = x^2$

Example $f(a.b) = a^2 + 2ab + b^2 = (a+b)^2$

check. $f(a.b) = \underset{1\times2}{(b \;\; a)} \underset{2\times2}{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}} \underset{2\times1}{\begin{pmatrix} a \\ b \end{pmatrix}} \rightarrow 1\times1 \quad = \vec{x}^T A \vec{x}$

$= (b \;\; a) \begin{pmatrix} a+b \\ a+b \end{pmatrix} = b(a+b) + a(a+b) = a^2+2ab+b^2$

Diagonalize. $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $\lambda_1 = 2$, $x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. $\lambda_2 = 0$. $x_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$X = [x_1 . \; x_2] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $\qquad X^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ (X is orthogonal!! $\underline{X^{-1} = X^T}$)

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \end{pmatrix} \begin{pmatrix} 2 & \\ & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \end{pmatrix}$$

!!!

$$= 2 \times \boxed{\underset{2\times1}{\begin{pmatrix} \frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} \end{pmatrix}} \overset{1\times2}{\left(\frac{1}{\sqrt2} . \frac{1}{\sqrt2}\right)}} + 0 \times \boxed{\begin{pmatrix} \frac{1}{\sqrt2} \\ -\frac{1}{\sqrt2} \end{pmatrix} \left(\frac{1}{\sqrt2} . -\frac{1}{\sqrt2}\right)}$$

$= \begin{pmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{pmatrix}$

projec matrix to a vector $\vec{x}$

$\frac{\vec{x}\,\vec{x}^T}{(\vec{x}\cdot\vec{x})}$ (if $\vec{x}$ is unit vecta $P = \vec{x}.\vec{x}^T$)

$x^T(A+B)x = x^TAx + x^TBx$

real number

$x^TAx = \sum \lambda_i \, x^T x_i \, x_i^T x$

$= \sum \lambda_i (x_i^T x)^2$

$f(a.b) = (a. \; b) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix}$ ("write quadratic function to sum of squares!!!)

$$= 2 \cdot (a.b) \boxed{\begin{pmatrix} \frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} \end{pmatrix} \left(\frac{1}{\sqrt2} , \frac{1}{\sqrt2}\right)} \begin{pmatrix} b \\ a \end{pmatrix} + 0 \times (a.b) \boxed{\begin{pmatrix} \frac{1}{\sqrt2} \\ -\frac{1}{\sqrt2} \end{pmatrix} \left(\frac{1}{\sqrt2} . -\frac{1}{\sqrt2}\right)} \begin{pmatrix} a \\ b \end{pmatrix}$$

$= \frac{1}{\sqrt2}a + \frac{1}{\sqrt2}b \quad = \frac{1}{\sqrt2}a + \frac{1}{\sqrt2}b \quad = \frac{1}{\sqrt2}a - \frac{1}{\sqrt2}b \quad = \frac{1}{\sqrt2}a - \frac{1}{\sqrt2}b$

$$= 2 \cdot \left(\frac{1}{\sqrt2}a + \frac{1}{\sqrt2}b\right)^2 + 0 \cdot \left(\frac{1}{\sqrt2}a - \frac{1}{\sqrt2}b\right)^2$$

Thm. $A = \lambda_1 \underset{\text{projection to } x_1}{x_1 x_1^T} + \lambda_2 \underset{\text{projection to } x_2}{x_2 x_2^T} + \cdots + \lambda_n \underset{\text{projection to } x_n}{x_n x_n^T}$ $\leftarrow$

and $x_1, x_2, \cdots, x_n$ are orthogonal!

For General $n \times n$ Symmetric Matrix A

eigenvalue $(\lambda_1, x_1) \cdots (\lambda_n, x_n)$

- $x_1 \cdots x_n$ are all orthogonal

(same calculation)

- $A = \lambda_1 \underbrace{x_1 x_1^T}_{\text{project to vector } x_1} + \lambda_2 \underbrace{x_2 x_2^T}_{\text{project to vector } x_2} + \cdots + \lambda_n \underbrace{x_n x_n^T}_{\text{project to vector } x_n}$

lemma $Ax_1 = \lambda_1 \cdot x_1$, $Ax_2 = \lambda_2 x_2$. A symmetric

$\lambda_1 \neq \lambda_2$, Aim. $x_1^T x_2 = 0$ (x_1, x_2 are orthogonal)

Calculate $x_1^T A x_2$

- $x_1^T A x_2 = x_1^T (\lambda_2 x_2)$ x_2 is the eigenvector.

$= \lambda_2 (x_1^T x_2)$

- $\underset{1\times n}{x_1^T} \underset{n\times n}{A} \underset{n\times 1}{x_2} = (x_1^T A x_2)^T$. $x_1^T A x_2$ is a real number

$= x_2^T A^T x_1$ $(AB)^T = B^T A^T$

$= x_2^T A x_1$ $A^T = A$ (A is symmetric)

(see ✡) ← $= x_2^T (\lambda x_1)$ x_1 is the eigenvector

$= \lambda_1 (x_1^T x_2)$

So! $\underbrace{\lambda_1 (x_1^T x_2) = \lambda_2 (x_1^T x_2)} = x_1^T A x_2$ $\lambda_1 \neq \lambda_2$

$\Rightarrow x_1^T x_2 = 0$!!

This tells us $x_1 \cdots x_n$ are orthogonal

if $\|x_1\|, = \cdots = \|x_n\| = 1$ are all unit vector

$\mathbb{X} = [x_1 \cdots x_n]$ is orthogonal ①

$A = \mathbb{X} \Lambda \mathbb{X}^{-1} = Q \Lambda Q^T$ ②

↙ orthogonal ↘ $\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$A = \sum \lambda_i \underbrace{x_i x_i^T}_{\text{projection}}$ ③

Thm. $A = \lambda_1 \underbrace{x_1 x_1^T}_{\substack{\text{projection} \\ \text{to } x_1}} + \lambda_2 \underbrace{x_2 x_2^T}_{\substack{\text{projection} \\ \text{to } x_2}} + \cdots + \lambda_n \underbrace{x_n x_n'}_{\substack{\text{projection} \\ \text{to } x_n}}$!! A is symmetric

and $x_1, x_2, \cdots, x_n$ are orthogonal ! !! Most important

lemma. $Ax_1 = \lambda_1 x_1$, $Ax_2 = \lambda_2 x_2$ and $\lambda_1 \neq \lambda_2$

Then $x_1^T x_2 = 0$

proof. Cachlate $x_1^T A x_2$

$x_1^T \to 1\times n$, $A \to n\times n$, $x_2 \to n\times 1$ } ⇒ a real number

- $x_1^T A x_2 = x_1^T (\lambda_2 x_2)$ because x_2 is eigenvector

$$= \lambda_2 (x_1^T x_2)$$

- $x_1^T A x_2 = (x_1^T A x_2)^T$ This is because $x_1^T A x_2$ is a real number.

$= x_2^T A^T x_1$ by the rule of transport of product

$= x_2^T A x_1$ A is symmetric

$= x_2^T (\lambda_1 x_1)$ $Ax_1 = \lambda x_1$

$= \lambda_1 (x_1^T x_2)$ $x_2^T x_1 = x_1^T x_2$ because it's dot product

We know $\lambda_1 (x_1^T x_2) = x_1^T A x_2 = \lambda_2 (x_1^T x_2)$

$\Rightarrow x_1^T x_2 = 0$ (for $\lambda_1 \neq \lambda_2$)

$X = [\underbrace{x_1 \cdots x_n}_{\text{all eigenvector}}] \Rightarrow X$ is an orthogonal matrix

$A = X \Lambda X^{-1} = Q \Lambda Q^T$ → Q is orthognal

Positive Definite Matrix. (P.S.D Matrix)

- PSD Matrix A means.
    - A is symmetric
    - all the eigenvalues $\lambda_i > 0$

$x_1 \cdots x_n$ are real numbers but variables.

$$\downarrow$$

$$\left(\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right)$$

why PSD Matrix?

$\vec{x}^T A \vec{x}$ is a qudratic function respect to $x_1 \cdots x_n$

$$x^T A x = \sum \underbrace{\lambda_i}_{\text{This is the Coefcient!}} \underbrace{(x_i^T x)^2}_{\text{This is square function}}$$

(eg. $(a+b)^2$, $(a-b)^2$)

always larger than 0

P.S.D Matrix $\Leftrightarrow$ it is sum of positive square function!!

So $x^T A x$ is always larger than zero (!)! $\leftrightarrow$ The same by eigen decomposition/digonalization

- If A and B are P.S.D Matrix. Then A+B is P.S.D.
- If A is P.S.D. Then A^{-1} is P.S.D.

① $x^T A x$. $x^T B x$ is always positive, so $x^T (A+B) x = x^T A x + x^T B x$ are also positive.

② if $A = A^T \Rightarrow (A^{-1})^T = (A^T)^{-1} \overset{A \text{ is symmetric}}{=} A^{-1} \Rightarrow A^{-1}$ is symmetric

if $Ax = \lambda x \Leftrightarrow A^{-1}x = \frac{1}{\lambda} x$

x is A's eigenvalue $\Leftrightarrow$ $\overset{\neq 0}{\lambda^{-1}}$ is A^{-1}'s eigenvalue.

A is symmetric. $x_1, x_2 \cdots x_n$ are orthog—l

$$A = \lambda_1 x_1 x_1^T + \lambda_2 x_2 x_2^T + \cdots + \lambda_n x_n x_n^T$$

$$A^{-1} = \lambda_1^{-1} x_1 x_1^T + \lambda_2^{-1} x_2 x_2^T + \cdots + \lambda_n^{-1} x_n x_n^T$$

Strang Section 6.4 – Symmetric Matrices and Section 6.5 – Positive Definite Matrices

Course notes adapted from N. Hammoud’s NYU lecture notes.

Symmetric Matrices

Diagonalizing a Symmetric Matrix

An $n \times n$ matrix A is symmetric if $A^T = A$.

The eigenvalues of a symmetric matrix are real and the eigenvectors are orthogonal (or can be made orthogonal).

Every symmetric matrix is diagonalizable

$$A = X\Lambda X^{-1}$$

eigenvectors are orthogonal
they can be made orthonormal

$$\implies A = Q\Lambda Q^T$$

orthogonal matrix: $Q^{-1} = Q^T$

Eigenvectors of a Symmetric Matrix

Let $\vec{x}_1, \vec{x}_2$ be eigenvectors of A associated with λ_1, λ_2, such that $\lambda_1 \neq \lambda_2$

$$\implies A\vec{x}_1 = \lambda_1\vec{x}_1, \quad A\vec{x}_2 = \lambda_2\vec{x}_2$$

(by def of e-value/e-vector)

We want to show that $\vec{x}_1 \perp \vec{x}_2 \implies \vec{x}_1^T\vec{x}_2 = 0$

Consider $\underbrace{\lambda_1 \vec{x}_1^T \vec{x}_2} = (x_1\lambda_1)^T \vec{x}_2 = (A\vec{x}_1)^T \vec{x}_2 = \vec{x}_1^T \underbrace{A^T \vec{x}_2} = \vec{x}_1^T \lambda_2 \vec{x}_2 = \underbrace{\lambda_2 \vec{x}_1^T \vec{x}_2}$

$\downarrow$

$\lambda_1^T = \lambda_1$

def of e-value

$= A\vec{x}_2$ (A is symmetric)

$= \lambda_2\vec{x}_2$

Thus $\lambda_1 \vec{x}_1^T \vec{x}_2 = \lambda_2 \vec{x}_1^T \vec{x}_2 \iff \lambda_1 \vec{x}_1^T \vec{x}_2 - \lambda_2 \vec{x}_1^T \vec{x}_2 = 0$

$(\lambda_1 - \lambda_2)\,\underline{\vec{x}_1^T \vec{x}_2} = 0$

$(\lambda_1 \neq \lambda_2)$ Thus $\boxed{\vec{x}_1^T \vec{x}_2 = 0}$

Example

Diagonalize $A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$. (into the form $Q \Lambda Q^T$ since A is symmetric)

- eigenvalues

$$|A - \lambda I| = \begin{vmatrix} 6-\lambda & -2 & -1 \\ -2 & 6-\lambda & -1 \\ -1 & -1 & 5-\lambda \end{vmatrix} = \cdots = (6-\lambda)(\lambda-8)(\lambda-3) = 0$$

$$\Rightarrow \lambda = 3, 6, 8$$

$\boxed{\lambda_1 = 3}$ $\text{Nul}(A - 3I)$

$$\left(\begin{array}{ccc|c} 3 & -2 & -1 & 0 \\ -2 & 3 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} \boxed{-1} & -1 & 2 & 0 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

x_3 : free

$x_2 = x_3$

$-x_1 - x_2 + 2x_3 = 0$

$x_1 = 2x_3 - x_3 = x_3$

$$\boxed{\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

Example

$\lambda_2 = 6$ $\quad \mathrm{Nul}(A - 6I)$:

$$\left(\begin{array}{ccc|c} 0 & -2 & -1 & 0 \\ -2 & 0 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right) \xrightarrow[\cdots]{} \vec{x}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

$\lambda_3 = 8$ $\quad \mathrm{Nul}(A - 8I)$:

$$\left(\begin{array}{ccc|c} -2 & -2 & -1 & 0 \\ -2 & -2 & -1 & 0 \\ -1 & -1 & -3 & 0 \end{array}\right) \longrightarrow \vec{x}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$A = \overset{Q}{\begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & 2/\sqrt{6} & 0 \end{pmatrix}} \overset{\Lambda}{\begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 8 \end{pmatrix}} \overset{Q^T \ (= Q^{-1})}{\begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}}$$

Positive Definite Matrices

Definition

An $n \times n$ matrix A is positive definite if:

(i) $A = A^T$

(ii) $\lambda_i > 0$ for all $1 \leq i \leq n$

The following statements are equivalent to "all eigenvalues are positive":

(1) all pivots are positive

(2) all upper left determinants are positive

(3) $\vec{x}^T A\vec{x}$ is positive for all $\vec{x} \neq 0$

Example

Show all equivalent positive definite properties for $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$.

- $\vec{x}^T A \vec{x} > 0 \quad \forall \vec{x} \neq \vec{0}$

these, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; $\vec{x}^T = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$

$$\vec{x}^T A \vec{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 & -x_1 + 2x_2 - x_3 & -x_2 + 2x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= 2\left(x_1^2 - x_1x_2 + x_2^2 - x_2x_3 + x_3^2\right) \quad > 0 \; ?$$

not clear yet

$x_1^2 + x_1^2 \qquad x_2^2 + x_2^2 \qquad x_3^2 + x_3^2$

$$= 2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + 2x_3^2$$

$$= \underset{\geq 0}{x_1^2} + \underset{\geq 0}{(x_1 - x_2)^2} + \underset{\geq 0}{(x_2 - x_3)^2} + \underset{\geq 0}{x_3^2}$$

but $x_1 \neq 0$ or $x_2 \neq 0$ or $x_3 \neq 0$

Properties

Theorem: If A is positive definite, then so is A^{-1}.

A is positive definite $\iff$ all of its eigenvalues are positive

Lemma: $\underbrace{\lambda \neq 0 \text{ is an e-value of } A}_{A\vec{x} = \lambda\vec{x} \text{ for some } \lambda \neq 0;\ \vec{x} \in \mathbb{R}^n} \iff \underbrace{\frac{1}{\lambda} \text{ is an e-value of } A^{-1}}_{A^{-1}\vec{v} = \frac{1}{\lambda}\vec{v} \text{ (for some } \vec{v} \in \mathbb{R}^n)}$

$(A \in M_{n\times n}(\mathbb{R}))$

pf (of lemma):

$$\begin{aligned} A\vec{x} = \lambda\vec{x} &\iff A^{-1}A\vec{x} = A^{-1}\lambda\vec{x} \\ &\iff \vec{x} = \lambda A^{-1}\vec{x} \\ &\iff A^{-1}\vec{x} = \frac{1}{\lambda}\vec{x} \end{aligned}$$

(same e-vector!)

<u>pf (of thm)</u>: A is positive definite, so all of its eigenvalues $\lambda_i > 0$ $(1 \leq i \leq n)$. So $\frac{1}{\lambda_i} > 0$ ($\lambda_i \neq 0$ $\forall i$, since A is invertible) and the e-values of A^{-1} are $\frac{1}{\lambda_i}$; so A^{-1} is pos. definite

Properties

Theorem: If A, B are positive definite, then $A+B$ is positive definite.

pf/ A and B are positive definite, so

$$\vec{x}^T A \vec{x} > 0 \quad \text{and} \quad \vec{x}^T B \vec{x} > 0 \quad \forall \vec{x} \in \mathbb{R}^n; \vec{x} \neq \vec{0}$$

(note: $\vec{x}$ is arbitrary, so we can keep it the same in both quadratic forms)

Adding both sides of the inequalities:

$$\vec{x}^T A \vec{x} + \vec{x}^T B \vec{x} > 0 + 0$$

which means $\vec{x}^T (A+B) \vec{x} > 0$

thus $A+B$ is positive definite

Properties

Theorem: If $A,\ B$ are positive definite, then $A + B$ is positive definite.