Linear Algebra

Lecture 19

Symmetric and Positive Definite Matrices

Dr. Ralph Chikhany

Strang Section 6.4 – Symmetric Matrices and Section 6.5 – Positive Definite Matrices

Course notes adapted from N. Hammoud's NYU lecture notes.

Symmetric Matrices

Diagonalizing a Symmetric Matrix

An $n \times n$ matrix A is symmetric if $A^T = A$.

The eigenvalues of a symmetric matrix are real and the eigenvectors are orthogonal (or can be made orthogonal).

Every symmetric matrix is diagonalizable

$$A = X \Lambda X^{-1}$$

eigenvectors are orthogonal
they can be made orthonormal

$$\implies A = Q \Lambda Q^T$$

orthogonal matrix: $Q^{-1} = Q^T$

Eigenvalues of a Symmetric Matrix

Eigenvectors of a Symmetric Matrix

Let $\vec{x}_1, \vec{x}_2$ be eigenvectors of A associated with λ_1, λ_2, such that $\lambda_1 \neq \lambda_2$

$$\implies A\vec{x}_1 = \lambda_1 \vec{x}_1, \quad A\vec{x}_2 = \lambda_2 \vec{x}_2$$

We want to show that $\vec{x}_1 \perp \vec{x}_2 \quad \implies \vec{x}_1^T \vec{x}_2 = 0$

Example

Diagonalize $A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$.

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Positive Definite Matrices

Definition

An $n \times n$ matrix A is positive definite if:

(i) $A = A^T$

(ii) $\lambda_i > 0$ for all $1 \leq i \leq n$

The following statements are equivalent to "all eigenvalues are positive":

(1) all pivots are positive

(2) all upper left determinants are positive

(3) $\vec{x}^T A \vec{x}$ is positive for all $\vec{x} \neq 0$

Example

Show all equivalent positive definite properties for $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$.

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Properties

Theorem: If A is positive definite, then so is A^{-1}.

Properties

Theorem: If $A,\ B$ are positive definite, then $A + B$ is positive definite.