NYU

Linear Algebra

Lecture 18

Diagonalization

Dr. Ralph Chikhany

Recap

Eigenvalue and Eigenvector

$$\underset{\text{square matrix}}{A}\ \underset{\text{eigenvector}}{\vec{x}} = \underset{\text{eigen value}}{\lambda}\ \vec{x}$$

① if $A x = \vec{0}$ or $x \in Nul(A)$, x is an eigenvector with eigenvalue 0

Whether A have 0 eigenvalue $\Leftrightarrow$ $\det(A) = 0$

② How to solve eigenvalue.

once we know λ, x is solved by $x \in Nul(A - \lambda I)$ Nullspace

x is the eigenvector with eigenvalue λ means $(A - \lambda I)x = \vec{0}$

λ is eigenvalue A $\Leftrightarrow$ $\det(A - \lambda I) = 0$

③ $p(\lambda) := \det(A - \lambda I)$

Fact 1. $p(\lambda)$ is a n-th order polynomial by cofactor Expansion

Fact 2. eigenvalue is the solution of. $p(\lambda) = 0$

Quiz. what is the eigenvalue of upper Traingular Matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & & \vdots \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ & a_{22}-\lambda & & \vdots \\ & & \ddots & \\ & & & a_{nn}-\lambda \end{bmatrix}$$

$\det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda)$

all eigenvalues of A is $a_{11} \cdot a_{22} \cdots a_{nn}$.

Fact 3 $n \times n$ matrix will have n eigenvalue $\lambda_1 \cdots \lambda_n$.

$$p(\lambda) = (-1)^n (\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$$

$(\ (\lambda - \lambda_1) \cdots (\lambda - \lambda_n) = \det(\lambda I - A) \ )$

$(\lambda I - A) = -(A - \lambda I)$

Why eigen are important`

$A^2x = A \cdot Ax = A \cdot (\lambda x) = \lambda Ax = \lambda \cdot \lambda x = \lambda^2 x$ ($\lambda \in \mathbb{R}$)

$Ax = \lambda x$ $\quad A^2x = \lambda^2 x, \quad A^3x = \lambda^3 x, \cdots, \quad A^k x = \lambda^k x$

(λ_i, x_i) $i = 1, 2, \cdots, n$ are all the eigenvalue and eigenvectors of A.

Now we have $\vec{x} = c_1 \vec{x_1} + c_2 \vec{x_2} + \cdots + c_n \vec{x_n}$

What is my $A\vec{x}$

$$A\vec{x} = A(c_1 \vec{x_1} + c_2 \vec{x_2} + \cdots + c_n \vec{x_n})$$
$$= c_1 A\vec{x_1} + c_2 A\vec{x_2} + \cdots + c_n A\vec{x_n}$$
$$= c_1\lambda_1 \vec{x_1} + c_2\lambda_2 \vec{x_2} + \cdots + c_n\lambda_n \vec{x_n}$$

$$A^k\vec{x} = c_1\lambda_1^k \vec{x_1} + c_2\lambda_2^k \vec{x_2} + \cdots + c_n\lambda_n^k \vec{x_n}$$

Thm. If A's eigenvalue $\lambda_1, \lambda_2, \cdots, \lambda_n$ are distinct.
Then $x_1 \cdots x_n$ are linear independent. (Not Required)

Start with $n=2$.

Basic idea. $A\vec{x}_1 = \lambda_1 \vec{x}_1, \quad A x_2 = \lambda_2 \vec{x_2}$

$\underbrace{c_1 \vec{x_1}}_{y_1} + \underbrace{c_2 \vec{x_2}}_{y_2} = \vec{0}$ (1) want to show $c_1 = c_2 = 0$.

$\lambda_1 \underbrace{c_1 \vec{x_1}}_{y_1} + \lambda_2 \underbrace{c_2 \vec{x_2}}_{y_2} = \vec{0}$ Times a A to the left of 1

$\begin{cases} y_1 + y_2 = 0 \\ \lambda_1 y_1 + \lambda_2 y_2 = 0 \end{cases}$ $\begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}$ is invertible if and only if $\lambda_1 \neq \lambda_2$.

$\Rightarrow \vec{y_1} = \vec{y_2} = \vec{0} \Rightarrow c_1 = c_2 = 0$.

General Case. $A\vec{x_1} = \lambda_1 \vec{x_1}$. $A\vec{x_2} = \lambda_2 \vec{x_2}, \cdots$ $A\vec{x_n} = \lambda_n \vec{x_n}$

$\underbrace{c_1 \vec{x_1}}_{\vec{y_1}} + \underbrace{c_2 \vec{x_2}}_{\vec{y_2}} + \cdots + \underbrace{c_n \vec{x_n}}_{\vec{y_n}} = 0$ want to show $c_1 = \cdots = c_n = 0$

$\lambda_1 c_1 \vec{x_1} + \lambda_2 c_2 \vec{x_2} + \cdots + \lambda_n c_n \vec{x_n} = 0$ Times A

$\lambda_1^2 c_1 \vec{x_1} + \lambda_2^2 c_2 \vec{x_2} + \cdots + \lambda_n^2 c_n \vec{x_n} = 0$ Times A^2

$\vdots$

$\lambda_1^{n-1} c_1 \vec{x_1} + \lambda_2^{n-1} c_2 \vec{x_2} + \cdots + \lambda_n^{n-1} c_n \vec{x_n} = 0$ Time A^{n-1}

$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & & & \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix}$ is invertible if and only if

$\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

Thm. If A's eigenvalue $\lambda_1, \lambda_2, \cdots, \lambda_n$ are distinct.
Then $x_1 \cdots x_n$ are linear independent $\Rightarrow$ $\underbrace{X = [x_1 \cdots x_n]}_{\text{is invertible!}}$

$$AX = A[x_1 \cdots x_n] = [Ax_1 \cdots Ax_n] = [\lambda_1 x_1 \ \ \lambda_2 x_2 \cdots \lambda_n x_n]$$

Check (Δ)

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 a_{11} & \lambda_2 a_{12} \\ \lambda_1 a_{21} & \lambda_2 a_{22} \end{bmatrix} = \underbrace{[x_1 \cdots x_n]}_{X}\underbrace{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}}_{\text{diag Matrix}} \quad (\Delta)$$

latex \Lambda

$AX = X\Lambda$, where Λ is diag Matrix $\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

$\longrightarrow$ $A = X \Lambda X^{-1}$ diagonalization!! (X is invertible)

Remark. - Not every matrix have a diagonalization!!

- If A have distinct eigenvalues, Then A can be diagonalized.

- If A can't be diagnolize, means A have two same eigenvalue. $\dim(Nul(A-\lambda I)) \geq 2$

$A = X\Lambda X^{-1}$. What is A^k

$$A^2 = A \cdot A = X \Lambda \underbrace{X^{-1} X}_{I} \Lambda X^{-1}$$

Λ is diag

$$= X \Lambda \Lambda X^{-1} = X \Lambda^2 X^{-1}$$

$$\Lambda^2 = \begin{bmatrix} \lambda_1^2 & & & \\ & \lambda_2^2 & & \\ & & \ddots & \\ & & & \lambda_n^2 \end{bmatrix}$$

$$A^k = X \Lambda^k X^{-1}$$

$$\Lambda^k = \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{bmatrix}$$

Similar Matrix

$A = X B X^{-1}$, A and B are similar Matrix

If A can be diagonalized means A is similar to $\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

① $\det(A) = \det(B)$

② A and B have the same eigenvalues

① $\det(A) = \det(X B X^{-1})$

$\det(X)\cdot\det(X^{-1}) = \det(I) = 1$

$X X^{-1} = I$

$= \det(X)\det(B)\det(X^{-1}) = \det(B)$

② if A has an eigen-vector x

$Ax = \lambda \cdot x$ $\quad X B X^{-1} x = \lambda \cdot x \Rightarrow B X^{-1} x = \lambda X^{-1} x$

x in "A" $\quad X^{-1}x$ in "B"

Prop 1. $\lambda_1 \cdots \lambda_n = \det(A)$

$A = X \Lambda X^{-1}$ $\quad \det(A) = \det(\Lambda) = \lambda_1 \cdots \lambda_n$ (Not Rigorous)

Prop 2 $\lambda_1 + \cdots + \lambda_n = a_{11} + \cdots + a_{nn}$.

call sum of the diagonal term "trace" of the matrix

$\operatorname{tr}(A)$

lemma. $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ is right even when A and B are triangluar

example $A = \begin{pmatrix} a \\ b \end{pmatrix}_{2\times1}$ $\quad B = (x, y)$

$\operatorname{tr}(AB) = \operatorname{tr}\left(\underset{2\times1}{\begin{pmatrix} a \\ b \end{pmatrix}} \underset{1\times2}{(x, y)}\right) = \operatorname{tr}\left(\underset{2\times2}{\begin{pmatrix} ax & ay \\ bx & by \end{pmatrix}}\right) = ax + by$

$\operatorname{tr}(BA) = \operatorname{tr}\left(\underset{1\times2}{(x, y)} \underset{2\times1}{\begin{pmatrix} a \\ b \end{pmatrix}}\right) = \operatorname{tr}\left(\underset{1\times1}{(xa + yb)}\right) = ax + by$

Example $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

$$tr\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right) = Tr\left(\begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & \\ & a_{21}b_{12}+a_{22}b_{22} \end{pmatrix}\right)$$

$$= a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{22}$$

$$tr\left(\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) = Tr\left(\begin{pmatrix} b_{11}a_{11}+b_{12}a_{21} & \\ & b_{21}a_{12}+b_{22}a_{22} \end{pmatrix}\right)$$

$$= b_{11}a_{11} + \underset{ij}{b_{12}}\,\underset{ji}{a_{21}} + \underset{ij}{b_{21}}\,\underset{ji}{a_{12}} + b_{22}a_{22}$$

$$\sum_{\text{possible } (i,j)} b_{ij}a_{ji}$$

Once we know $Tr(AB) = Tr(BA)$

why $tr(A) = \lambda_1 + \cdots + \lambda_n$

$$tr(A) = tr(X \Lambda X^{-1})$$

$$= tr(\Lambda \underbrace{X^{-1} X}_{I})$$

$$= tr(\Lambda) = \lambda_1 + \cdots + \lambda_n$$

Remark.

$$Tr(ABC) \neq Tr(CBA)$$
$$\neq$$
$$Tr(ACB)$$

$$Tr(ABC) = Tr(CAB)$$

$$Tr(ABC) = Tr(BCA)$$

Strang Section 6.2 – Diagonalizing a Matrix

Course notes adapted from N. Hammoud's NYU lecture notes.

Diagonalization

Diagonalizing a Matrix? What and Why?

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \xrightarrow{\text{diagonalize}} \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

\lambda

\lambda

e-values of A

Diagonalization is useful if we want to compute powers of A.

$A^k \to$ hard

$\Lambda^k \to$ easy

we may be interested in finding what a matrix does to a vector if it acts on it over and over again

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}; \quad A^{\cancel{3}\,32} = \underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdots \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}_{32 \text{ times!}}$$

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}; \quad D^2 = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}; \quad D^3 = \begin{pmatrix} 8 & 0 \\ 0 & 27 \end{pmatrix} \cdots D^n = \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix}$$

since D is diagonal

Diagonalization Theorem

Let $A \in \mathbb{M}_{n\times n}(\mathbb{R})$. Then A is diagonalizable if and only if

A as n linearly independent eigenvectors.

Corollary: If a matrix has n distinct eigenvalues, then it is diagonalizable.

⚠ If a matrix is diagonalizable, it does not necessarily have n distinct eigenvalues. Diagonalizability is related to linear independence of e-vectors.

⚠ Diagonalizability and invertibility are not interchangeable

How to Diagonalize a Matrix

start:

$A = ? \Lambda ?$

$A = X \Lambda X^{-1}$

e-values

We want: $X^{-1}AX = \Lambda$

X^{-1} exists iff cols of X are linearly independent

Thus, A is diagonalizable only if X^{-1} exists.

It turns out that $X = [\vec{x_1}\ \vec{x_2}\ \ldots\ \vec{x_n}]$ contains the eigenvectors of A associated with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Why is $X^{-1}AX = \Lambda$?

Let A be a matrix with e-values $\{\lambda_1, \lambda_2 \ldots \lambda_n\}$ (some possibly repeated) and $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ the e-vectors associated with $\lambda_1, \lambda_2, \ldots, \lambda_n$ (respectively)

$$AX = A\underbrace{\begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix}}_{\text{e-vectors}} = \begin{bmatrix} A\vec{x}_1 & A\vec{x}_2 & \cdots & A\vec{x}_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1\vec{x}_1 & \lambda_2\vec{x}_2 & \cdots & \lambda_n\vec{x}_n \end{bmatrix}$$

$$X\Lambda = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} = \begin{bmatrix} \vec{x}_1\lambda_1 & \vec{x}_2\lambda_2 & \cdots & \vec{x}_n\lambda_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1\vec{x}_1 & \lambda_2\vec{x}_2 & \cdots & \lambda_n\vec{x}_n \end{bmatrix}$$

So $AX = X\Lambda \implies X^{-1}AX = X^{-1}X\Lambda \implies \boxed{X^{-1}AX = \Lambda}$

or $\iff \boxed{A = X\Lambda X^{-1}}$

Why is $X^{-1}AX = \Lambda$?

Thus

$$A = X \Lambda X^{-1}$$

$\to$ inverse of X

X: eigenvectors in column form

Λ: eigenvalues on the diagonals, 0 everywhere else

⚠ X and Λ store e-vectors and e-values in respective order

(this is a factorization of A, aka the diagonalization of A)

Order of Eigenvalues and Eigenvectors

Note: The order of the eigenvectors in X must be the same a the order of the eigenvalues in Λ.

$$\text{If } X = [\vec{x}_1 \ \vec{x}_2 \ \dots \ \vec{x}_n], \text{ then } \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$\text{If } X = [\vec{x}_2 \ \vec{x}_1 \ \dots \ \vec{x}_n], \text{ then } \Lambda = \begin{bmatrix} \lambda_2 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

Powers of A

If A is diagonalizable, then $A = X\Lambda X^{-1}$. we get:

$$A^2 = (X\Lambda X^{-1})(X\Lambda X^{-1}) = (X\Lambda)\underbrace{(X^{-1}X)}(\Lambda X^{-1})$$

(matrix multiplication is associative)

$$= (X\Lambda)\ I\ (\Lambda X^{-1})$$

$$= X\Lambda\Lambda X^{-1} = X\Lambda^2 X^{-1}$$

$$A^4 = X\Lambda\underbrace{X^{-1}X}_{I}\Lambda\underbrace{X^{-1}X}_{I}\Lambda\underbrace{X^{-1}X}_{I}\Lambda X^{-1} = X\Lambda^4 X^{-1}$$

$\vdots$

$$\boxed{A^n = X\Lambda^n X^{-1}}$$

$$\begin{pmatrix} \lambda_1^n & & & \\ & \lambda_2^n & & \\ & & \ddots & \\ & & & \lambda_n^n \end{pmatrix}$$

n multiplications

3 multiplications, since Λ^n can be easily found

Example

Let $A = \begin{bmatrix} 7 & -2 \\ 4 & 1 \end{bmatrix}$. Is A diagonalizable? If yes, diagonalize it.

In general:

$n \times n$ matrix $\longrightarrow$ n distinct e-values? $\xrightarrow{\text{yes}}$ Diagonalizable! :)

n distinct e-values? $\xrightarrow{\text{no}}$ check e-vectors $\longrightarrow$ n linearly independent $\xrightarrow{\text{yes}}$ Diagonalizable!

n linearly independent $\xrightarrow{\text{no}}$ " not diagonalizable

Here, A is a 2×2 matrix. we first find the e-values

$$|A-\lambda I| = \begin{vmatrix} 7-\lambda & -2 \\ 4 & 1-\lambda \end{vmatrix} = (7-\lambda)(1-\lambda)+8$$

$$= 7-8\lambda+\lambda^2+8 = \lambda^2-8\lambda+15$$

$$p(\lambda) = (\lambda-3)(\lambda-5)$$

We have 2 distinct e-values: $\lambda=3$ and $\lambda=5$

$\Rightarrow A$ <u>is</u> diagonalizable

Example

Let $A = \begin{bmatrix} 7 & -2 \\ 4 & 1 \end{bmatrix}$. Is A diagonalizable? If yes, diagonalize it.

Next, find the e-vectors

$\lambda = 3$ $\quad \mathrm{Nul}(A - 3I)$; $\mathrm{Nul}\begin{pmatrix} 4 & -2 \\ 4 & -2 \end{pmatrix}$; $\left(\begin{array}{cc|c} 4 & -2 & 0 \\ 4 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

x_2 is free, $x_1 = \frac{1}{2} x_2$ $\quad$ so $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = c \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $(c \in \mathbb{R})$

$\lambda = 5$ $\quad \mathrm{Nul}(A - 5I)$; $\mathrm{Nul}\begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix}$; $\left(\begin{array}{cc|c} 2 & -2 & 0 \\ 4 & -4 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

x_2 is free, $x_1 = x_2$ $\quad$ so $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Example

Let $A = \begin{bmatrix} 7 & -2 \\ 4 & 1 \end{bmatrix}$. Is A diagonalizable? If yes, diagonalize it.

Thus: $X = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \Rightarrow X^{-1} = \frac{1}{1-2} \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$ (using shortcut from lecture 16, for 2×2 matrices)

$X^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Thus: $A = X \Lambda X^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$

(or: $A = X \Lambda X^{-1} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}}_{\text{new } X} \underbrace{\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}}_{\text{new } \Lambda} \underbrace{\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}}_{\text{new } X^{-1}}$)

Example

Let $A = \begin{bmatrix} 7 & -2 \\ 4 & 1 \end{bmatrix}$. Is A diagonalizable? If yes, diagonalize it.

We can now generalize $A^n = X\Lambda^n X^{-1}$

$$A^n = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}^n \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

$$A^n = \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3^n & 0 \\ 0 & 5^n \end{bmatrix}}_{\begin{bmatrix} 3^n & 5^n \\ 2(3^n) & 5^n \end{bmatrix}} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

Thus $A^n = \begin{bmatrix} -3^n + 2(5^n) & 3^n - 5^n \\ -2(3^n) + 2(5^n) & 2(3^n) - 5^n \end{bmatrix}$

ex: A^{10} can be easily found now.

Another Example

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$: $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 0 \overset{\text{set}}{=} 0$

$(1-\lambda)^2 = 0 \Rightarrow \lambda = 1$

We have 1 eigenvalue for a 2x2 matrix, we are not guaranteed diagonalizability (yet). We check the e-space.

$\lambda = 1$ $\text{Nul}(A - 1I)$: $\left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ x_1 free, $x_2 = 0$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ 1e-span is span $\underbrace{\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}}_{\text{one generating vector (not 2)}}$ $\Rightarrow$ A is not diagonalizable

EFY: If possible, diagonalize

$A = \begin{bmatrix} 7 & 0 & -3 \\ -9 & -2 & 3 \\ 18 & 0 & -8 \end{bmatrix}$

Similar Matrices

When are two Matrices Similar?

$A = X\Lambda X^{-1}$ $\longrightarrow$ A and Λ are called similar matrices

In general, A and B are similar if

$$A = MBM^{-1}$$

for some invertible matrix M.

Note: · In general, A need not be diagonalizable and B need not be diagonal

· $A = MBM^{-1} \iff B = M^{-1}AM$

· Similar matrices have: same rank, same trace, same determinant and same characteristic polynomial (e-values) but different eigenvectors

However: two matrices having the same rank, same trace, same determinant and same eigenvalues does not necessarily imply that they're similar.

Similar Matrices Have the Same Characteristic Polynomial

let A and B be similar matrices. Then there is some matrix C such that $A = CBC^{-1}$

char. polyn of A is $|A-\lambda I|$, char polyn. of B is $|B-\lambda I|$

$$A-\lambda I = CBC^{-1} - \lambda I \qquad // \ \lambda I = \lambda CC^{-1}I = C(\lambda I)C^{-1}$$

$$= CBC^{-1} - C(\lambda I)C^{-1}$$

$$= C(BC^{-1} - (\lambda I)C^{-1}) = C(B-\lambda I)C^{-1}.$$

Thus

$$|A-\lambda I| = |C(B-\lambda I)C^{-1}| \qquad // \ |MN| = |M||N|$$

$$= \cancel{|C|}\,|B-\lambda I|\,\cancel{|C^{-1}|} \qquad // \ |C^{-1}| = 1/|C|$$

$$= |B-\lambda I|$$