Lecture 18

Diagonalization

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Strang Section 6.2 – Diagonalizing a Matrix

Course notes adapted from N. Hammoud's NYU lecture notes.

Diagonalization

Diagonalizing a Matrix? What and Why?

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \xrightarrow{\text{diagonalize}} \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

Diagonalization is useful if we want to compute powers of A.

$A^k \to$ hard

$\Lambda^k \to$ easy

we may be interested in finding what a matrix does to a vector if it acts on it over and over again

Diagonalization Theorem

Let $A \in \mathbb{M}_{n\times n}(\mathbb{R})$. Then A is diagonalizable if and only if

A as n linearly independent eigenvectors.

Corollary: If a matrix has n distinct eigenvalues, then it is diagonalizable.

How to Diagonalize a Matrix

We want: $X^{-1}AX = \Lambda$

Thus, A is diagonalizable only if X^{-1} exists.

It turns out that $X = [\vec{x_1}\ \vec{x_2}\ \dots\ \vec{x_n}]$ contains the eigenvectors of A associated with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$.

Why is $X^{-1}AX = \Lambda$?

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Order of Eigenvalues and Eigenvectors

Note: The order of the eigenvectors in X must be the same a the order of the eigenvalues in Λ.

$$\text{If } X = [\vec{x}_1 \ \vec{x}_2 \ \ldots \ \vec{x}_n], \text{ then } \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$\text{If } X = [\vec{x}_2 \ \vec{x}_1 \ \ldots \ \vec{x}_n], \text{ then } \Lambda = \begin{bmatrix} \lambda_2 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

Powers of A

Example

Let $A = \begin{bmatrix} 7 & -2 \\ 4 & 1 \end{bmatrix}$. Is A diagonalizable? If yes, diagonalize it.

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Another Example

Similar Matrices

When are two Matrices Similar?

$A = X\Lambda X^{-1} \quad \longrightarrow \quad A$ and Λ are called similar matrices

In general, A and B are similar if

$$A = MBM^{-1}$$

for some invertible matrix M.

Similar Matrices Have the Same Characteristic Polynomial