Lecture 17

Eigenvalues

Dr. Ralph Chikhany

Recap.

hint: $P = A(A^TA)^{-1}A^T$

1. P is a projection matrix Then $P^2 = P$

2. $\det(A) = 3$. $A \in \mathbb{R}^{n\times n}$, What is $\det(C)$

   C is the matrix of Cofactor. hint. $A^{-1} = \frac{1}{\det A} C^T$

hard

3. P is projection. $I - P$ is also Projection!

Q1. $P^2 = \underbrace{A(A^TA)^{-1}A^T}_{=P}\;\underbrace{A(A^TA)^{-1}A^T}_{=P} = A(A^TA)^{-1}A^T = P$

What does $P^2 = P$ mean?

① $P^2x = Px$ for every vector x (we are considering P^2 and P are the same linear Transform)

② $Py = y$ for every $y \in Col(P)$

$y \in Col(P) \iff \{y \mid y = Px, \}$

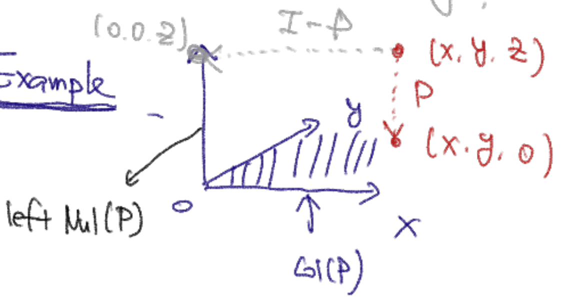
Remark $P^2 = P \not\Rightarrow$ projection in our text book.

This is because P is a projection, then $P^2 = P$

Q3. $I - P$ $\quad (I-P)^2 = I - 2P + P^2 = I - 2P + P = I - P$

$-\ (I-P)y = y$, if y lies in $I - P$

Example



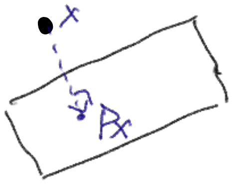
$P(x,y,z) = (x,y,0)$

$P(x,y,0) = (x,y,0)$ (checking $P^2 = P$)

$(I-P)(x,y,z) = (x,y,z) - (x,y,0)$

$= (0,0,z)$

$Col(I-P) = \text{leftMul}(P)$



Col (P)

$(x - Px) \perp \mathrm{Col}(P)$

$\Leftrightarrow \quad x - Px \in$ left Nul (P)

- This means $(I - P)$ is the Projection that Project back to left Nul space.

$\Rightarrow \quad \underbrace{x -}_{} \underbrace{(I - P)\,x}_{} \perp$ left Nul (P) $\Leftarrow$ I want to show

$Px \in \mathrm{Col}(P)$, so $P(x) \perp$ left Nul (P)

2. $\det(A) = 3$. $A \in \mathbb{R}^{n \times n}$, What is $\det(C)$

C is the matrix of Cofactor. hint $A^{-1} = \frac{1}{\det A} C^T$

$\det(A^{-1}) = \det\left(\frac{1}{\det A} C^T\right) = \det\left(\frac{1}{\det A} C\right)$ Transport not changing determinate.

$= \frac{1}{\det(A)}$ This is because $A^{-1} A = I$

$\Rightarrow \det(A^{-1}) \cdot \det(A)$

$= \det(I) = 1$

$\Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$.

$= \frac{1}{\det(A)^n} \det(C)$ This using the property

$\det(cA) = c^n \det(A)$

$\Rightarrow \frac{1}{\det(A)} = \frac{1}{\det(A)^n} \det(C) \Rightarrow \det(C) = \det(A)^{n-1}$.

Strang Sections 6.1 – Introduction to Eigenvalues

Course notes adapted from N. Hammoud's NYU lecture notes.

Introductory Example

Introductory Example

In a population of rabbits:

1. half of the newborn rabbits survive their first year; $S_{n+1} = \frac{1}{2} f_n$
2. of those, half survive their second year; $t_{n+1} = \frac{1}{2} S_n$
3. their maximum life span is three years; $f_{n+1} = 6 S_n + 8 t_n$
4. rabbits have $0, 6, 8$ baby rabbits in their three years, respectively.

If you know the population one year, what is the population the next year?

$$f_n = \text{first-year rabbits in year } n$$
$$s_n = \text{second-year rabbits in year } n$$
$$t_n = \text{third-year rabbits in year } n$$

Introductory Example

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3. their maximum life span is three years;
4. rabbits have $0, 6, 8$ baby rabbits in their three years, respectively.

If you know the population one year, what is the population the next year?

$$\begin{aligned} f_n &= \text{first-year rabbits in year } n \\ s_n &= \text{second-year rabbits in year } n \\ t_n &= \text{third-year rabbits in year } n \end{aligned}$$

$V_n = \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix}$

The rules say:

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix} = \begin{pmatrix} f_{n+1} \\ s_{n+1} \\ t_{n+1} \end{pmatrix}.$$

$f_{n+1} = 6 s_n + 8 t_n$

$s_{n+1} = \frac{1}{2} f_n$

$t_{n+1} = \frac{1}{2} s_n$

Let $A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$ and $v_n = \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix}$. Then $\boxed{A v_n = v_{n+1}}$.

$V_{n+1} = A V_n$

$n+2$ year: $V_{n+2} = A V_{n+1}$

$= A \cdot (A V_n) = A^2 V_n$.

$n+3$ year: $V_{n+3} = A^3 V_n$

$n+4$ year: $V_{n+4} = A^4 V_n$

Introductory Example

today

in 10 years

The rules say:

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix} = \begin{pmatrix} f_{n+1} \\ s_{n+1} \\ t_{n+1} \end{pmatrix}.$$

Let $A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$ and $v_n = \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix}$. Then $\boxed{Av_n = v_{n+1}}$.

first year, second year, third year

v_0	v_{10}	v_{11}
$\begin{pmatrix} 3 \\ 7 \\ 9 \end{pmatrix}$	$\begin{pmatrix} 30189 \\ 7761 \\ 1844 \end{pmatrix}$	$\begin{pmatrix} 61316 \\ 15095 \\ 3881 \end{pmatrix}$
$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 9459 \\ 2434 \\ 577 \end{pmatrix}$	$\begin{pmatrix} 19222 \\ 4729 \\ 1217 \end{pmatrix}$
$\begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix}$	$\begin{pmatrix} 28856 \\ 7405 \\ 1765 \end{pmatrix}$	$\begin{pmatrix} 58550 \\ 14428 \\ 3703 \end{pmatrix}$

$\vec v_{10}$ (in 10 years)

depends on $\vec v_8$

$$\vec v_{10} = A \boxed{\vec v_9} = A \boxed{A \vec v_8} = \underbrace{AAA}_{A^3} \vec v_7 = \dots = A^{10} \vec v_0$$

depends on $\vec v_9$

Note: We can use this process to predict how the system evolves, and if the number of rabbits stabilizes at some level.

However: here, the system does not stabilize. We see that

$\vec v_{n+1} = \boxed{A \vec v_n = 2 \vec v_n}$ looks like $A\vec x = \lambda \vec x$

eigenvalue (eventually, can find eigenvector)

Eigenvalues and Eigenvectors

Recall

- What happens when a square matrix acts on a vector?
  - The vector is stretched
  - The vector is shrunk
  - The vector is rotated

Eigenvectors and Eigenvalues

- What happens when a square matrix acts on a vector?
  - The vector is stretched
  - The vector is shrunk
  - The vector is rotated

"proportion of different kind rabits will not change"

- There are vectors known as eigenvectors of a square matrix, such that when the matrix acts on such vectors, they remain in the same direction.

Matrix scalar vector

$$A\vec{x} = \lambda\vec{x}$$

If $|\lambda| = 1 \implies$ the magnitude of $\vec{x}$ is unchanged
If $|\lambda| < 1 \implies \vec{x}$ is shrunk
If $|\lambda| > 1 \implies \vec{x}$ is stretched

always the same size

A should always be a square matrix

$\vec{x}$ is eigen vector

$A\vec{x} = \lambda\vec{x}$

$A^2\vec{x} = A \cdot (\lambda\vec{x}) = \lambda A\vec{x}$
$= \lambda^2\vec{x}$

$\rightarrow A^3\vec{x} = \lambda^3\vec{x}$

$\vdots$

$A^n\vec{x} = \lambda^n\vec{x}$

Example

~~Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Are $\vec{u}$ and $\vec{v}$ eigenvectors of A?~~

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Elimination Method will change the eigenvals

$EA = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $Au_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$\lambda_1 = 2$

$u_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $Au_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\lambda_2 = 0$

$u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $(EA)u_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_1 = 1$

$u_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $(EA)u_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\lambda_2 = 0$.

- A is not invertible. (A is square matrix)

  $Ax = 0$ will have some non-trivial solution

  $x \in Nul(A)$ are eigenvectors, with eigen value 0.

- A have 0 eigenvalue $\Leftrightarrow$ $\det(A) = 0$

Magic Thing!

- $n \times n$ matrix have n eigen.
- $\lambda_1 + \lambda_2 + \cdots + \lambda_n = a_{11} + \cdots + a_{nn}$

Example

Find a basis for the 2-eigenspace of

λ

$$A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}.$$

eigenspace: space generated by the eigenvectors to a corresponding eigenvalue

$\lambda = 2$ is an e-value $\iff A\vec{x} = 2\vec{x}$ for some vector(s) $\vec{x}$

$\iff A\vec{x} - 2\vec{x} = \vec{0}$

$\iff (A-2I)\vec{x} = \vec{0}$

Thus, finding e-vectors (or e-space) comes down to finding the sol. set to the equation $\underbrace{(A-2I)}_{\text{matrix}}\vec{x} = \vec{0}$, or $\boxed{\text{Nul}(A-2I)}$

$$A - 2I = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{pmatrix} \rightarrow$$

Find its nulspace

Example

Find a basis for the 2-eigenspace of

λ

$$A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}.$$

eigenspace: space generated by the eigenvectors to a corresponding eigenvalue

$$\begin{pmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{pmatrix} \sim \begin{array}{c} \begin{pmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{array}{ccc} & x_2 & x_3 \end{array} \end{array}$$

x_2, x_3 free

$2x_1 = x_2 - 6x_3 \Rightarrow x_1 = \frac{1}{2}x_2 - 3x_3$

Solution set (e-vectors)

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x_2 - 3x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

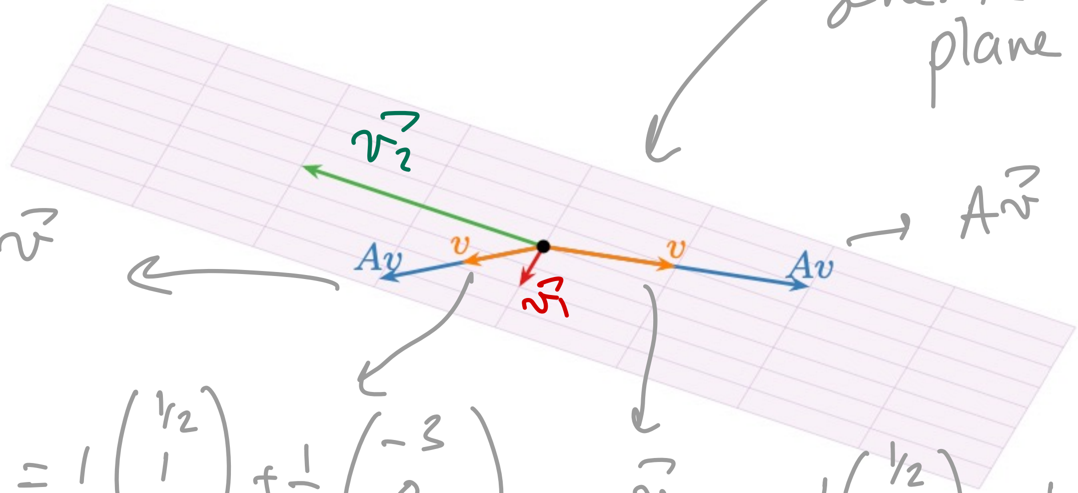
2-eigenspace of A is the plane generated by $\left\{ \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$, containing the origin $\rightarrow$ e-spaces are subspaces

Example

$\vec{v}_1 \qquad \vec{v}_2$

A basis for the 2-eigenspace of $\begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}$ is $\underbrace{\left\{ \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}}_{}$.

generate this plane



$A\vec{v} = 2\vec{v}$

$A\vec{v} = 2\vec{v}$

$\vec{v} = 1\begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$

$\vec{v} = -1\begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$

Example

Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Are $\vec{u}$ and $\vec{v}$ eigenvectors of A?

i.e. Do $\vec{u}$ and $\vec{v}$ verify the equation $A\vec{x} = \lambda\vec{x}$?

(check if $A\vec{u} = \underset{\lambda \in \mathbb{R}}{\underline{\lambda}}\vec{u}$ and $A\vec{v} = \underset{\mu \in \mathbb{R}}{\underline{\mu}}\vec{v}$)

$$A\vec{u} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \end{bmatrix} \neq \lambda\begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ for any } \lambda$$

$\vec{u}$ is not an e-vector of A

$$A\vec{v} = \underbrace{\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{A\vec{v}} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= 2 \cdot \vec{v}$$

$\vec{v}$ is an e-vector associated with e-value $\lambda = 2$

Example

Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$. Show that $\lambda = 7$ is an eigenvalue of A, and find the corresponding eigenvector.

Example

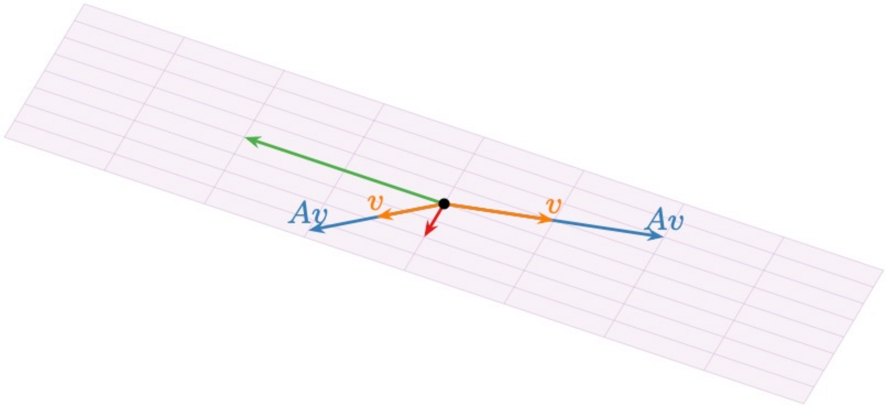
Find a basis for the 2-eigenspace of

λ

$$A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}.$$

Example

A basis for the 2-eigenspace of $\begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}$ is $\left\{ \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$.



Computing Eigenvalues and Eigenvectors

Summary

Let $A \in \mathbb{M}_{n\times n}(\mathbb{R})$

- An eigenvector of A is a non-zero vector $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \lambda\vec{x}$ for some $\lambda \in \mathbb{R}$.
- An eigenvalue of A is a value $\lambda \in \mathbb{R}$ such that the equation $A\vec{x} = \lambda\vec{x}$ has a nontrivial solution. In this case, we say λ is the eingenvalue associated with eigenvectos $\vec{x}$.

How to Compute Eigenvalues and Eigenvectors

$$A\vec{x} = \lambda\vec{x} \iff A\vec{x} = \lambda I\vec{x} \iff A\vec{x} - \lambda I\vec{x} = \vec{0}$$

eigenvalue

$$\iff \boxed{(A - \lambda I)\vec{x} = \vec{0}}$$

λ is an eigenvalue of A

$\iff \mathrm{Nul}(A - \lambda I)$ is not $\{\vec{0}\}$

$\iff A - \lambda I$ is not invertible

$\iff \det(A - \lambda I) = 0$

deal this as a function, variable is λ.

Therefore, the eigenvector $\vec{x}$ is in $\mathrm{Nul}(A - \lambda I)$.

Recall that $\vec{x} \neq \vec{0}$

- If $\vec{x} = \vec{0}$ is the only solution, then $(A - \lambda I)$ has linearly independent columns, i.e., no free columns, then the matrix $(A - \lambda I)$ is invertible.

- If $\vec{x} \neq \vec{0}$, then the matrix $(A - \lambda I)$ has free columns, i.e., it is not invertible.

characteristic polynomial

$$\implies \boxed{\det(A - \lambda I) = 0}$$

as a function of λ

is a polynomial of λ.

Example – Find the Characteristic Polynomial

Example. $A = \begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$ $\quad A - \lambda I = \begin{pmatrix} 0-\lambda & 6 & 8 \\ 1/2 & 0-\lambda & 0 \\ 0 & 1/2 & 0-\lambda \end{pmatrix} = \begin{pmatrix} -\lambda & 6 & 8 \\ 1/2 & -\lambda & 0 \\ 0 & 1/2 & -\lambda \end{pmatrix} \leftarrow$

using cofactor expansion!

$$\det(A-\lambda I) = 0 \cdots + (-1)^{2+3}\left(\tfrac{1}{2}\right) \times \begin{vmatrix} -\lambda & 8 \\ 1/2 & 0 \end{vmatrix} + (-1)^{3+3} \times (-\lambda) \times \begin{vmatrix} -\lambda & 6 \\ 1/2 & -\lambda \end{vmatrix}$$

$$= -1/2 \times (0 - 8 \times 1/2) + \overbrace{(-\lambda) \times \underbrace{((-\lambda)\times(-\lambda) - 6 \times 1/2)}_{\hookrightarrow\ n-1 \text{ order polynoiml.}}}^{n\text{-th order}}$$

$$= 2 - \lambda(\lambda^2 - 3) = \underbrace{-\lambda^3 - 3\lambda + 2}_{P(\lambda)}.$$

More general.

we induction! if $(n-1)\times(n-1)$ matrix's $P(\lambda)$ is $(n-1)$-th order polynoiml $\Rightarrow$ it's also true for $n \times n$

- $P(\lambda)$ of a $n \times n$ Matrix is a n-th order polynoimal.

In General

More general:

- $P(\lambda)$ of a $n \times n$ Matrix is a n-th order polynomial.

we induction! if $(n-1)\times(n-1)$ matrix's $P(\lambda)$ is $(n-1)$-th order polynomial $\Rightarrow$ it's also true for $n \times n$

- all the eigen-values λ of a matrix is the solution of

$$P(\lambda) = 0.$$

---

a $n \times n$ matrix will have n eigenvalues / eigenvectors

$\Downarrow$

$\lambda_1 \cdots \lambda_n$

☆ $P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n).$

Examples

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \Rightarrow P(\lambda) = \begin{vmatrix} 5-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix}$$

$\lambda_1 = 3 - 2\sqrt{2}$

$\lambda_2 = 3 + 2\sqrt{2}$.

$$= (5-\lambda)(1-\lambda) - 2 \times 2$$

$$= \lambda^2 - 6\lambda + 1$$

$$= (\lambda - \underbrace{3 - 2\sqrt{2}}_{\lambda_1})(\lambda - \underbrace{3 + 2\sqrt{2}}_{\lambda_2}).$$

$ax^2 + bx + c = 0$

$\Rightarrow x = \frac{-4b \pm \sqrt{\Delta}}{2a}$

- $Ax = \lambda x$
- λ is eigen $\Leftrightarrow$ $\det(A - \lambda I) = 0$

$P(\lambda)$ is a n-th order polynomial $\rightarrow$ n - eigen values.

Solve $P(\lambda) = 0$ gives you eigenvalue.

Finding the eigenvalues and associated eigenvectors

Example

Let $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. Find the eigenvalues and eigenvectors.

- Eigenvalues: $p(\lambda) = |A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1 = \lambda^2 - 6\lambda + 9 - 1$

$= \lambda^2 - 6\lambda + 8$

$p(\lambda) = 0 \Rightarrow (\lambda - 2)(\lambda - 4) = 0 \to \lambda_1 = 2, \ \lambda_2 = 4$ (2 distinct eigenvalues)

- Eigenvectors:

$\lambda = 2$ $A\vec{x} = 2\vec{x} \Rightarrow (A - 2I)\vec{x} = \vec{0}$; Find Nul $(A - 2I)$

$A - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow[\text{Nul}]{} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ x_2 free, $x_1 = -x_2$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ so the 2-e-space is span$\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

e-vector

Example

Let $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. Find the eigenvalues and eigenvectors.

$\lambda = 4$

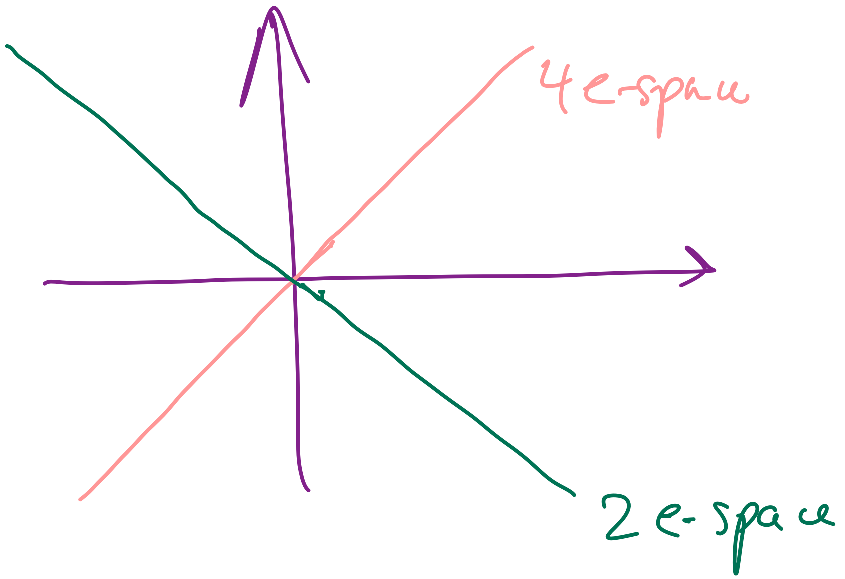
$A\vec{x} = \dots\vec{x} \Rightarrow (A - 4I)\vec{x} = \vec{0}$; Find Nul$(A-4I)$

$A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \xrightarrow[\text{Nul}]{} \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ x_2 free, $x_1 = x_2$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so the 4-e-space is span$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

e-vector

e-value	e-vector	e-space
$\lambda = 2$	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$	line $y = -x$
$\lambda = 4$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	line $y = x$



⚠ e-spaces are not always orthogonal

Example

Let $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$. Find the eigenvalues and eigenvectors.

Here, $|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^{2} - 0 \overset{\text{set}}{=} 0 \Rightarrow \underbrace{\lambda = 3}$

the only e-value with multiplicity 2

$\lambda = 3$ 3-espace

$A - 3I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ Nul: $\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ x_1 free, $x_2 = 0$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. So the 3-espace is the x-axis

an eigenvector

(shortage of e-vectors)

More on Eigenvalues

Trace and Determinant

Def. The trace of an $n \times n$ matrix A is the sum of the diagonal entries of A.

$$tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

Thm. If A is an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then

(i) $\det A = \lambda_1 \lambda_2 \ldots \lambda_n$

(ii) $tr(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$

Elimination does not Preserve Eigenvalues

Eigenvalues of a Triangular Matrix

Invertibility and Eigenvalues

Invertible Matrix Theorem:

A is invertible if and only if 0 is not an eigvenvalue of A

Linear Independence of Eigenvectors

If $\vec{v}_1, v_2, \ldots, v_n$ are eigenvectors of a matrix $A \in \mathbb{M}_{n\times n}(\mathbb{R})$

associated with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$

then $\{\vec{v}_1, \vec{v_2}, \ldots, \vec{v_n}\}$ is linearly independent.

Corollary: An $n \times n$ matrix has at most n distinct eigenvalues.