Lecture 17
Eigenvalues

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Strang Sections 6.1 – Introduction to Eigenvalues

Course notes adapted from N. Hammoud's NYU lecture notes.

Introductory Example

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In a population of rabbits:

1. half of the newborn rabbits survive their first year;
2. of those, half survive their second year;
3. their maximum life span is three years;
4. rabbits have $0, 6, 8$ baby rabbits in their three years, respectively.

If you know the population one year, what is the population the next year?

$$
\begin{aligned}
f_n &= \text{first-year rabbits in year } n \\
s_n &= \text{second-year rabbits in year } n \\
t_n &= \text{third-year rabbits in year } n
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The rules say:

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix} = \begin{pmatrix} f_{n+1} \\ s_{n+1} \\ t_{n+1} \end{pmatrix}.$$

Let $A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$ and $v_n = \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix}$. Then $Av_n = v_{n+1}$.

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v_0	v_{10}	v_{11}
$\begin{pmatrix} 3 \\ 7 \\ 9 \end{pmatrix}$	$\begin{pmatrix} 30189 \\ 7761 \\ 1844 \end{pmatrix}$	$\begin{pmatrix} 61316 \\ 15095 \\ 3881 \end{pmatrix}$
$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 9459 \\ 2434 \\ 577 \end{pmatrix}$	$\begin{pmatrix} 19222 \\ 4729 \\ 1217 \end{pmatrix}$
$\begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix}$	$\begin{pmatrix} 28856 \\ 7405 \\ 1765 \end{pmatrix}$	$\begin{pmatrix} 58550 \\ 14428 \\ 3703 \end{pmatrix}$

Eigenvalues and Eigenvectors

Recall

- What happens when a square matrix acts on a vector?
  - The vector is stretched
  - The vector is shrunk
  - The vector is rotated

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- What happens when a square matrix acts on a vector?
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- There are vectors known as eigenvectors of a square matrix, such that when the matrix acts on such vectors, they remain in the same direction.

$$A\vec{x} = \lambda\vec{x}$$

If $|\lambda| = 1 \implies$ the magnitude of $\vec{x}$ is unchanged

If $|\lambda| < 1 \implies \vec{x}$ is shrunk

If $|\lambda| > 1 \implies \vec{x}$ is stretched

Example

Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Are $\vec{u}$ and $\vec{v}$ eigenvectors of A?

Example

Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$. Show that $\lambda = 7$ is an eigenvalue of A, and find the corresponding eigenvector.

Example

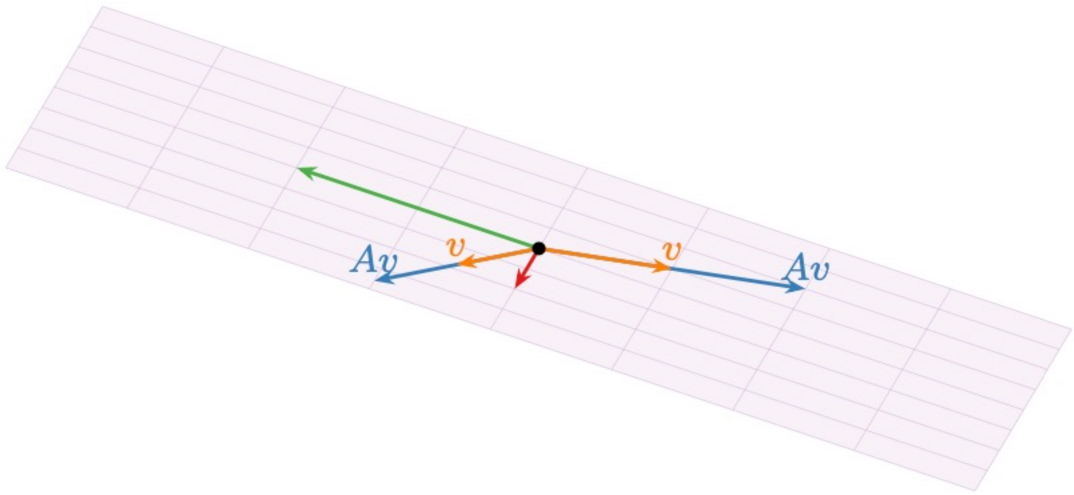
Find a basis for the 2-eigenspace of

λ

$$A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}.$$

Example

A basis for the 2-eigenspace of $\begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}$ is $\left\{ \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$.



Computing Eigenvalues and Eigenvectors

Summary

Let $A \in \mathbb{M}_{n\times n}(\mathbb{R})$

- An eigenvector of A is a non-zero vector $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \lambda\vec{x}$ for some $\lambda \in \mathbb{R}$.
- An eigenvalue of A is a value $\lambda \in \mathbb{R}$ such that the equation $A\vec{x} = \lambda\vec{x}$ has a nontrivial solution. In this case, we say λ is the eingenvalue associated with eigenvectos $\vec{x}$.

How to Compute Eigenvalues and Eigenvectors

$$A\vec{x} = \lambda\vec{x} \iff A\vec{x} = \lambda I\vec{x} \iff A\vec{x} - \lambda I\vec{x} = \vec{0}$$

$$\iff (A - \lambda I)\vec{x} = \vec{0}$$

Therefore, the eigenvector $\vec{x}$ is in $\text{Nul}(A - \lambda I)$.

Recall that $\vec{x} \neq \vec{0}$

- If $\vec{x} = \vec{0}$ is the only solution, then $(A - \lambda I)$ has linearly independent columns, i.e., no free columns, then the matrix $(A - \lambda I)$ is invertible.
- If $\vec{x} \neq \vec{0}$, then the matrix $(A - \lambda I)$ has free columns, i.e., it is not invertible.

characteristic polynomial

$$\implies \det(A - \lambda I) = 0$$

Example – Find the Characteristic Polynomial

In General

Examples

Finding the eigenvalues and associated eigenvectors

Example

Let $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. Find the eigenvalues and eigenvectors.

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More on Eigenvalues

Trace and Determinant

Def. The trace of an $n \times n$ matrix A is the sum of the diagonal entries of A.

$$tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

Thm. If A is an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then

(i) $\det A = \lambda_1 \lambda_2 \ldots \lambda_n$

(ii) $tr(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$

Elimination does not Preserve Eigenvalues

Eigenvalues of a Triangular Matrix

Invertibility and Eigenvalues

Invertible Matrix Theorem:

A is invertible if and only if 0 is not an eigvenvalue of A

Linear Independence of Eigenvectors

If $\vec{v}_1, v_2, \ldots, v_n$ are eigenvectors of a matrix $A \in \mathbb{M}_{n\times n}(\mathbb{R})$

associated with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$

then $\{\vec{v}_1, \vec{v_2}, \ldots, \vec{v_n}\}$ is linearly independent.

Corollary: An $n \times n$ matrix has at most n distinct eigenvalues.