NYU

Lecture 16
Cofactors

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Strang Sections 5.2 – Permutations and Cofactors a and Section 5.3 – Cramer's Rule, Inverses and Volumes

Course notes adapted from N. Hammoud's NYU lecture notes.

Cofactors

The Minor of an Element

Let A be an $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & & & & \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & & & & \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

→ $(n-1) \times (n-1)$ square

For any element a_{ij}, the minor M_{ij} is the determinant of the $(n-1) \times (n-1)$ submatrix that does not include the row and column containing a_{ij}.

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 3 & 7 & 13 \end{bmatrix}$. Compute the minor of a_{23}.

$\det\left[\begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}\right] = 1 \times 7 - 2 \times 3 = 1$

The Cofactor of an Element

Let A be an $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & & & & \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & & & & \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix}$$

The cofactor of any element a_{ij} is $C_{ij} = (-1)^{i+j} M_{ij}$.

$$C_{23} = (-1)^{2+3} M_{23}$$

$$= (-1) \times 1 = 1$$

Cofactor Expansion

Let A be an $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & & & & \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & & & & \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix}$$

The cofactor of any element a_{ij} is $C_{ij} = (-1)^{i+j} M_{ij}$.

To obtain the cofactor expansion of any row or column of A, we multiply each element in the row or column by its cofactor, and sum them up.

- Cofactor expansion along row 1:

  $a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$
- Cofactor expansion along column 2:

  $a_{12}C_{12} + a_{22}C_{22} + \cdots + a_{n2}C_{n2}$

all the element on the column 2)

For any $n \times n$ matrix, the cofactor expansion along any row or column equals det A.

Cofactor Method to Compute Determinants

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

smallest number of non-zeros

Use cofactor expansion to compute $\begin{vmatrix} 1 & 0 & 3 \\ -4 & 2 & 1 \\ -2 & 0 & 2 \end{vmatrix}$.

$$= 1 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} + 0 \cdot (-1)^{1+2} \begin{vmatrix} -4 & 1 \\ -2 & 2 \end{vmatrix} + 3 \cdot (-1)^{1+3} \begin{vmatrix} -4 & 2 \\ -2 & 0 \end{vmatrix}$$

$$= 4 + 0 + 3 \cdot 4 = 16$$

Cofactor expansion

$$2 \times (-1)^{2\times 2} \times \begin{vmatrix} 1 & 3 \\ -2 & 2 \end{vmatrix} = 2 \times (1 \times 2 - (-2) \times 3) = 16$$

Example

$$\begin{vmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ a & 0 & a & 0 & 3 \\ b & a & 0 & a & 0 \\ 0 & b & 0 & 0 & a \end{vmatrix}$$

Note: There are enough zeros here in some rows/cols, but we could get even more.

Recall: (row operations)
- $R_i \leftrightarrow R_j$ — det scales by (-1)
- $\alpha R_i \to R_i$ — det scales by α
- $\alpha R_i + R_j \to R_j$ — determinant does not change

Goal: use pivot ① and make all entries below it a zero.

$R_3 - aR_1$, $R_4 - bR_1$:

$$\begin{vmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & -2a & 0 & 3 \\ 0 & a & -3b & a & 0 \\ 0 & b & 0 & 0 & a \end{vmatrix} = 1 \begin{vmatrix} 1 & 0 & 3 & 0 \\ 0 & -2a & 0 & 3 \\ a & -3b & a & 0 \\ b & 0 & 0 & a \end{vmatrix} = -b \begin{vmatrix} 0 & 3 & 0 \\ -2a & 0 & 3 \\ -3b & a & 0 \end{vmatrix} + 0(\dots) - 0(\dots) + a \begin{vmatrix} 1 & 0 & 3 \\ 0 & -2a & 0 \\ a & -3b & a \end{vmatrix}$$

$$= \dots = \boxed{4a^3 + 27b^2}$$

Ask on CW for similar exercises

Example

$$\begin{vmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ a & 0 & a & 0 & 3 \\ b & a & 0 & a & 0 \\ 0 & b & 0 & 0 & a \end{vmatrix}$$

Cofactor Method to Compute the Inverse

Let A be an $n \times n$ matrix and let C be the matrix of cofactors of the elements of A. If $\det A \neq 0$, then

$$A^{-1} = \frac{1}{\det A} C^T$$

Compute the inverse of $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ using the cofactor method.

First, find each c_{ij} (for the cofactor matrix)

$$c_{11} = (-1)^{1+1} a_{22} = +a_{22}$$

$$c_{12} = (-1)^{1+2} a_{21} = -a_{21}$$

$$c_{21} = (-1)^{2+1} a_{12} = -a_{12}$$

$$c_{22} = (-1)^{2+2} a_{11} = +a_{11}$$

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$$c_{21} = (-1)^{2+1} a_{12} = -a_{12}$$

$$c_{22} = (-1)^{2+2} a_{11} = +a_{11}$$

$$C = \begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix} \Rightarrow C^T = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

$$\boxed{A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}}$$

(can use this shortcut)

Shortcut to Compute the Inverse of a 2×2 matrix

Compute the inverse of $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ using the cofactor method.

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Thm. $A^{-1} = \frac{1}{\det(A)} C^T.$

C is the matrix of the Cofactors!

Cramer's Rule

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This is a method that allows you to solve $A\vec{x} = \vec{b}$ using determinants.

Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

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$$\det(A) \cdot x_1 = \det\left(\begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}\right) \Rightarrow x_1 = \frac{\det\left(\begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}\right)}{\det(A)}$$

- To find x_1

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}$$

$$A \cdot \left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right] = \left[A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad A\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad A\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right]$$

$$A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = b, \quad A\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}, \quad A\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}$$

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Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

det A $\quad (-1)^{2+2} \cdot x_2 \cdot 1 \quad$ det(...)

- To find x_2

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix}$$

$$x_2 = \frac{\det\left(\begin{bmatrix} a_{12} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix}\right)}{\det(A)}$$

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This is a method that allows you to solve $A\vec{x} = \vec{b}$ using determinants.

Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

- To find x_3

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & x_1 \\ 0 & 1 & x_2 \\ 0 & 0 & x_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}$$

Example

$$(S):\begin{cases} 2x+2y+z=1 \\ 2x+y-z=2 \\ 3x+y+z=3 \end{cases}$$

$$\det(A)=\det\left(\begin{pmatrix} 2 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{pmatrix}\right)=-7$$

$$x=\frac{\det\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{pmatrix}}{\det(A)}=\frac{-9}{-7}$$

$$y=\frac{\det\begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 3 & 3 & 1 \end{pmatrix}}{\det(A)}=\frac{5}{-7}$$

$$z=\frac{\det\begin{pmatrix} 2 & 2 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 3 \end{pmatrix}}{\det(A)}=\frac{1}{-7}$$

Cramer's Rule is easy for you

is hard for Computer

For $n\times n$ matrix

- Cramer's Rule

  Computer $n!$ times

- LU decomposition

  Computer n^3 times

$$A^{-1} = \frac{1}{\det A} C^T$$

Solve $A A^{-1} = I$

$$A \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 \end{bmatrix}$$

Column Representation of A^{-1}

To Solve A^{-1} $\Leftrightarrow$ Solve n equations

$$A v_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad A v_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

If you use Cramer's Rule to solve them.

$$A^{-1} = \frac{1}{\det A} C^T$$ (need $n!$ computations for computers)

(elimination method. computation n^3)

Example

$$(S):\begin{cases} 2x+2y+z=1 \\ 2x+y-z=2 \\ 3x+y+z=3 \end{cases}$$