NYU

Linear Algebra

Lecture 16

Cofactors

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Strang Sections 5.2 – Permutations and Cofactors a and Section 5.3 – Cramer's Rule, Inverses and Volumes

Course notes adapted from N. Hammoud's NYU lecture notes.

Cofactors

The Minor of an Element

Let A be an $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & & & & \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & & & & \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix}$$

For any element a_{ij}, the minor M_{ij} is the determinant of the $(n-1) \times (n-1)$ submatrix that does not include the row and column containing a_{ij}.

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 3 & 7 & 13 \end{bmatrix}$. Compute the minor of a_{23}.

The Cofactor of an Element

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The cofactor of any element a_{ij} is $C_{ij} = (-1)^{i+j} M_{ij}$.

Cofactor Expansion

Let A be an $n \times n$ matrix

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The cofactor of any element a_{ij} is $C_{ij} = (-1)^{i+j} M_{ij}$.

To obtain the cofactor expansion of any row or column of A, we multiply each element in the row or column by its cofactor, and sum them up.

- Cofactor expansion along row 1:
  $a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$
- Cofactor expansion along column 2:
  $a_{12}C_{12} + a_{22}C_{22} + \cdots + a_{n2}C_{n2}$

For any $n \times n$ matrix, the cofactor expansion along any row or column equals $\det A$.

Cofactor Method to Compute Determinants

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

Use cofactor expansion to compute $\begin{vmatrix} 1 & 0 & 3 \\ -4 & 2 & 1 \\ -2 & 0 & 2 \end{vmatrix}$.

Example

$$\begin{vmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ a & 0 & a & 0 & 3 \\ b & a & 0 & a & 0 \\ 0 & b & 0 & 0 & a \end{vmatrix}$$

Shortcut to Compute the Inverse of a 2×2 matrix

Compute the inverse of $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ using the cofactor method.

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Cramer's Rule

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This is a method that allows you to solve $A\vec{x} = \vec{b}$ using determinants.

Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

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- To find x_1

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}$$

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- To find x_2

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix}$$

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- To find x_3

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & x_1 \\ 0 & 1 & x_2 \\ 0 & 0 & x_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}$$

Example

$$(S) : \begin{cases} 2x + 2y + z = 1 \\ 2x + y - z = 2 \\ 3x + y + z = 3 \end{cases}$$

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