NYU



# Lecture 2

# Vectors and Spans

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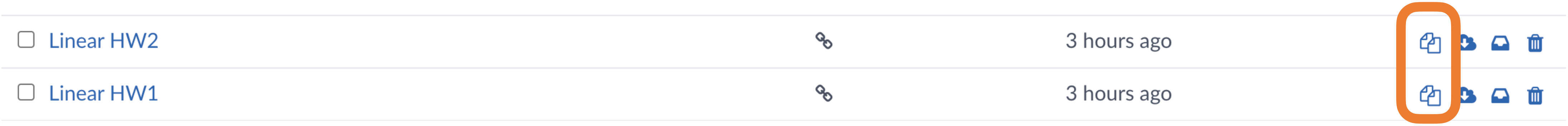
**Based on Dr. Ralph Chikhany's Slide**

# Reminders

- Get access to Gradescope, Campuswire.
- Obtain the textbook.
- Problem Set 1 due by 11.59 pm on Friday (NY time).
  - ✓ Late work policy applies.
- Recap Quiz 1 due by 11.59 pm on Sunday (NY time).
  - ❖ Late work policy does not apply.
- Recap Quiz is timed.
  - ❑ Once you start, you have 60 minutes to finish it (even if you close the tab)

Latex -> Overleaf -> Copy



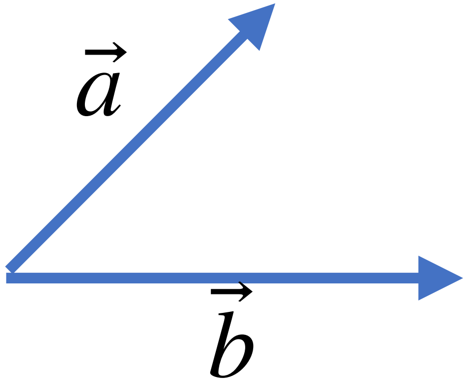

You can put what you want to recap in the (anonymous) form.

# ReCap

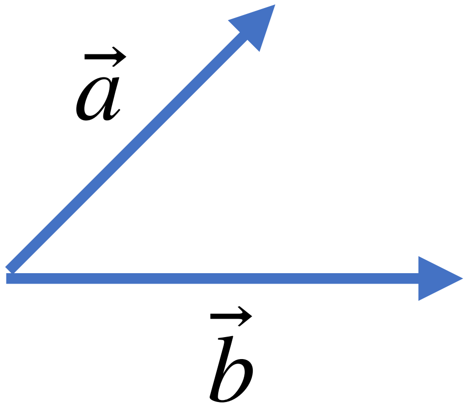
Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text

# Vector Addition

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$




$$\vec{a} + \vec{b} = ?$$



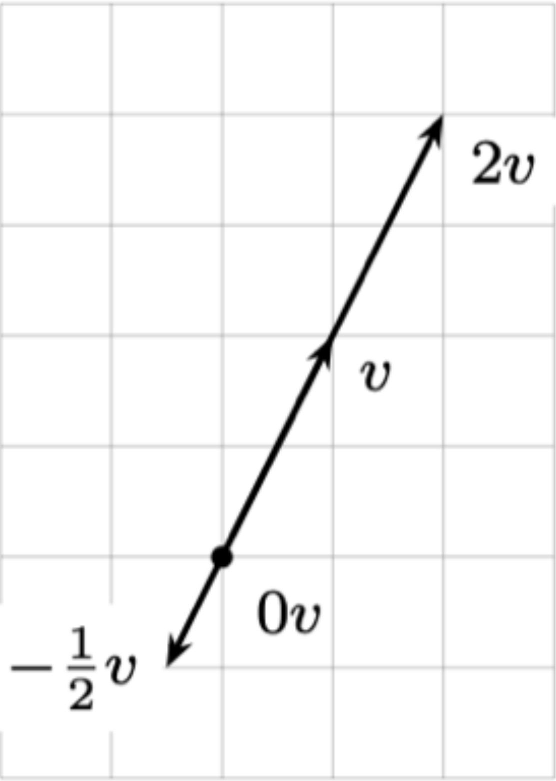

$$\vec{a} - \vec{b} = ?$$

$$\vec{a} - \vec{a} = ?$$

# Scalar vector multiplication

$$\textcolor{red}{c}\begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$

Some multiples of $v$.



# Dot Product

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$

Dot product is a linear combination

# Dot Product

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$

**Example 3** Dot products enter in economics and business. We have three goods to buy and sell. Their prices are $(p_1, p_2, p_3)$ for each unit—this is the "price vector" $\boldsymbol{p}$. The quantities we buy or sell are $(q_1, q_2, q_3)$—positive when we sell, negative when we buy. *Selling* $q_1$ *units at the price* $p_1$ *brings in* $q_1 p_1$. The total income (quantities $q$ times prices $p$) is ***the dot product*** $\boldsymbol{q} \cdot \boldsymbol{p}$ ***in three dimensions***:

$$\textbf{\textit{Income}} = (q_1, q_2, q_3) \cdot (p_1, p_2, p_3) = q_1 p_1 + q_2 p_2 + q_3 p_3 = \textbf{\textit{dot product.}}$$

# Dot Product

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$

Length

Distance

$\|\vec{v}\|$ And $|c|$

Unit Vector:

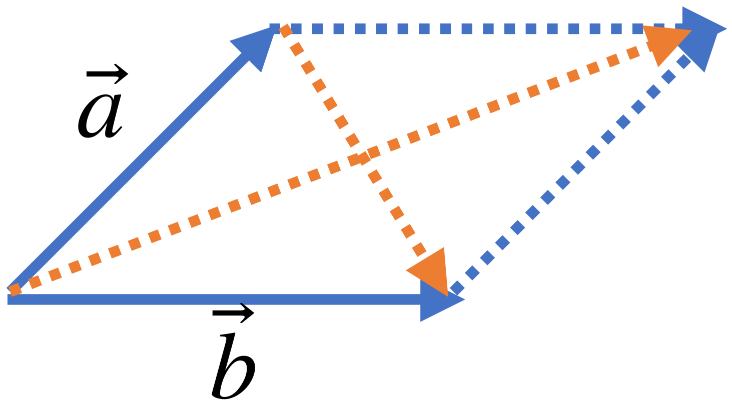
What is the unit vector of $(1,1)$?

# Dot Product

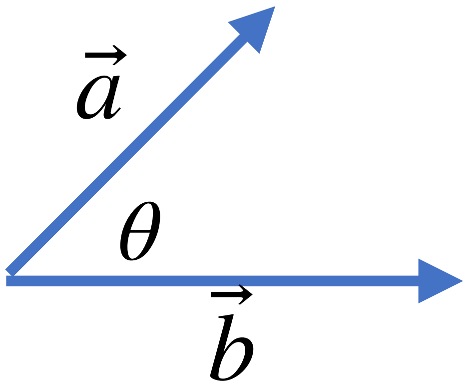
Communicative $\vec{a} \cdot \vec{b} =$

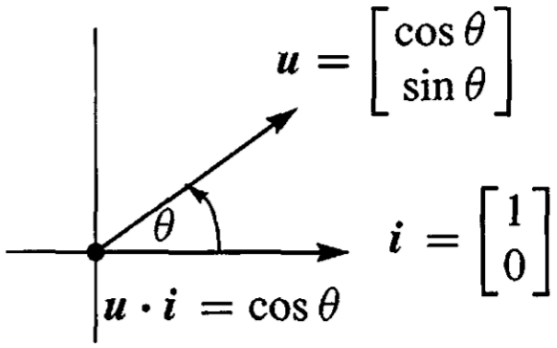
Distributive $(\vec{a} + \vec{b}) \cdot \vec{c} =$

**Example** $||\vec{a} + \vec{b}||^2 + ||\vec{a} - \vec{b}||^2 =$



# Angle

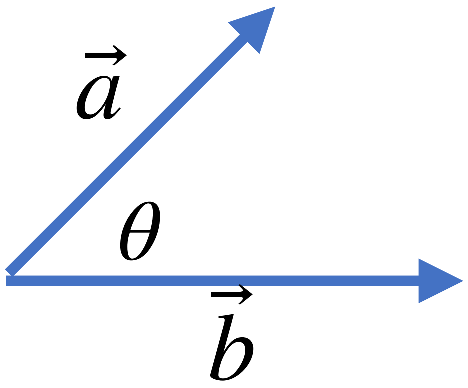


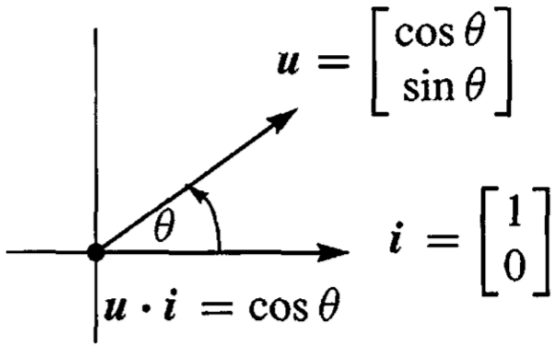



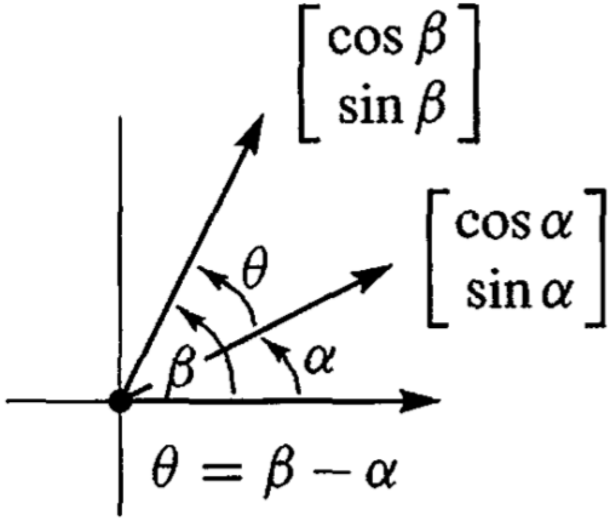

How to decide whether $\theta$ is larger than $\frac{\pi}{2}$

What is the unit vector parallel/orthogonal to $(4,3)$?

# Angle



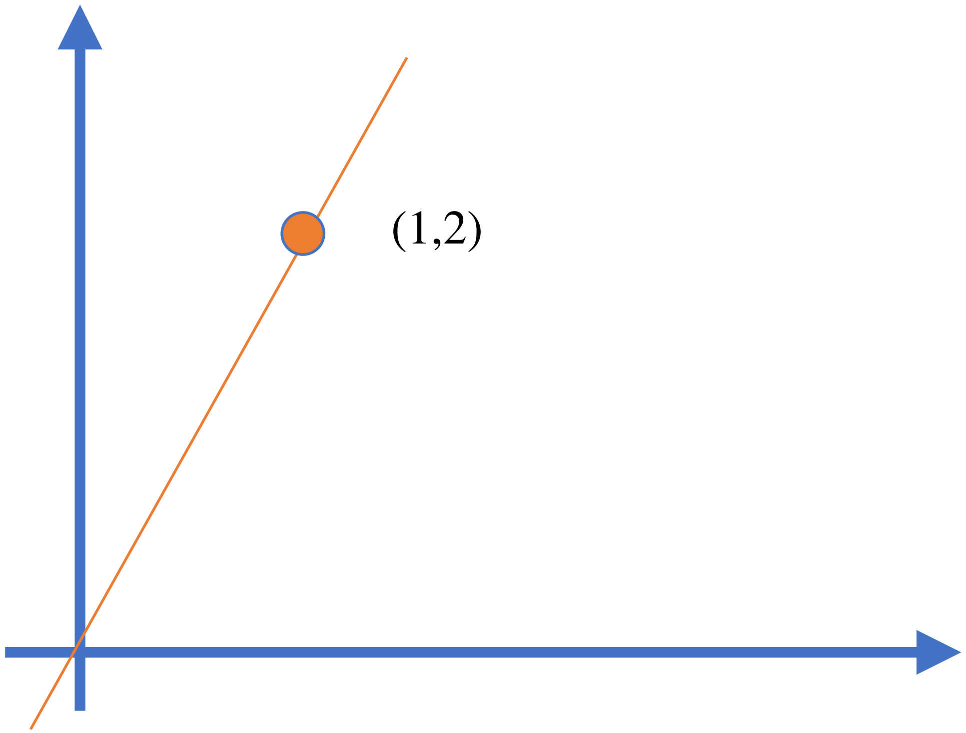

calculate $\cos(\beta - \alpha)$

# Motivation: Best fit of linear equation

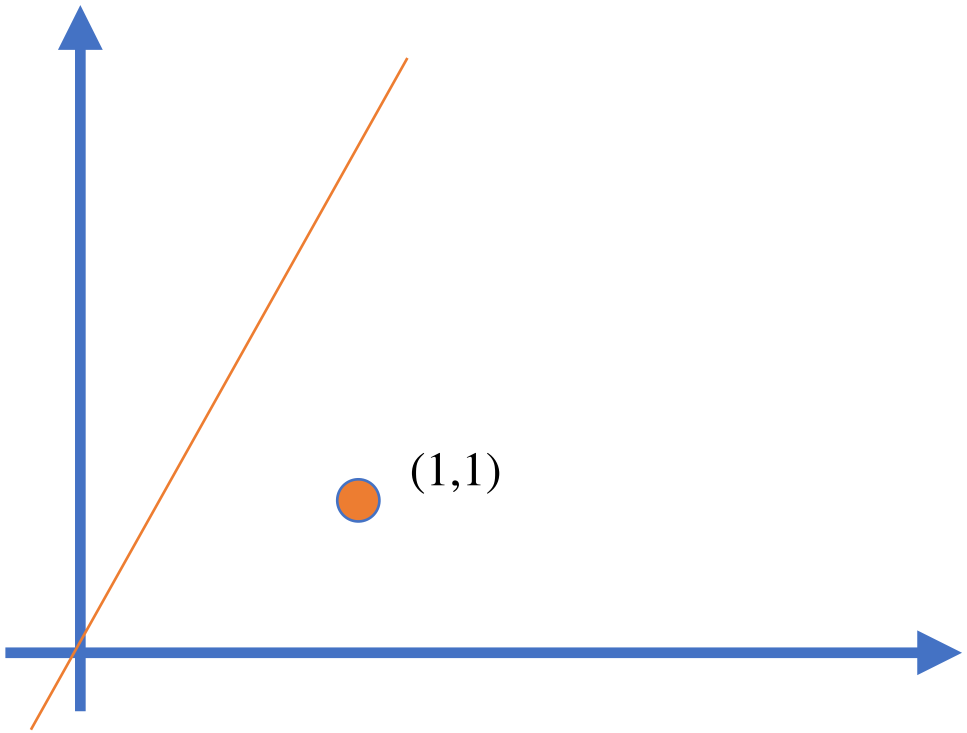
Not Required

overdetermined linear system

$$2x = 2$$
$$x = 1$$




$$2x = 1$$
$$x = 1$$



# Example

**1.2 C** Find a vector $\boldsymbol{x} = (c, d)$ that has dot products $\boldsymbol{x} \cdot \boldsymbol{r} = 1$ and $\boldsymbol{x} \cdot \boldsymbol{s} = 0$ with the given vectors $\boldsymbol{r} = (2, -1)$ and $\boldsymbol{s} = (-1, 2)$.

How is this question related to Example **1.1 C**, which solved $c\boldsymbol{v} + d\boldsymbol{w} = \boldsymbol{b} = (1, 0)$?

**1.1 C** Find two equations for the unknowns $c$ and $d$ so that the linear combination $c\boldsymbol{v} + d\boldsymbol{w}$ equals the vector $\boldsymbol{b}$:

$$\boldsymbol{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \boldsymbol{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

# Inequalities

**SCHWARZ INEQUALITY** $|\boldsymbol{v} \cdot \boldsymbol{w}| \leq \|\boldsymbol{v}\| \, \|\boldsymbol{w}\|$

**TRIANGLE INEQUALITY** $\|\boldsymbol{v} + \boldsymbol{w}\| \leq \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$

**Example 6** The dot product of $\boldsymbol{v} = (a, b)$ and $\boldsymbol{w} = (b, a)$ is $2ab$. Both lengths are $\sqrt{a^2 + b^2}$. The Schwarz inequality in this case says that $2ab \leq a^2 + b^2$.

# Reminder: Linear Combination

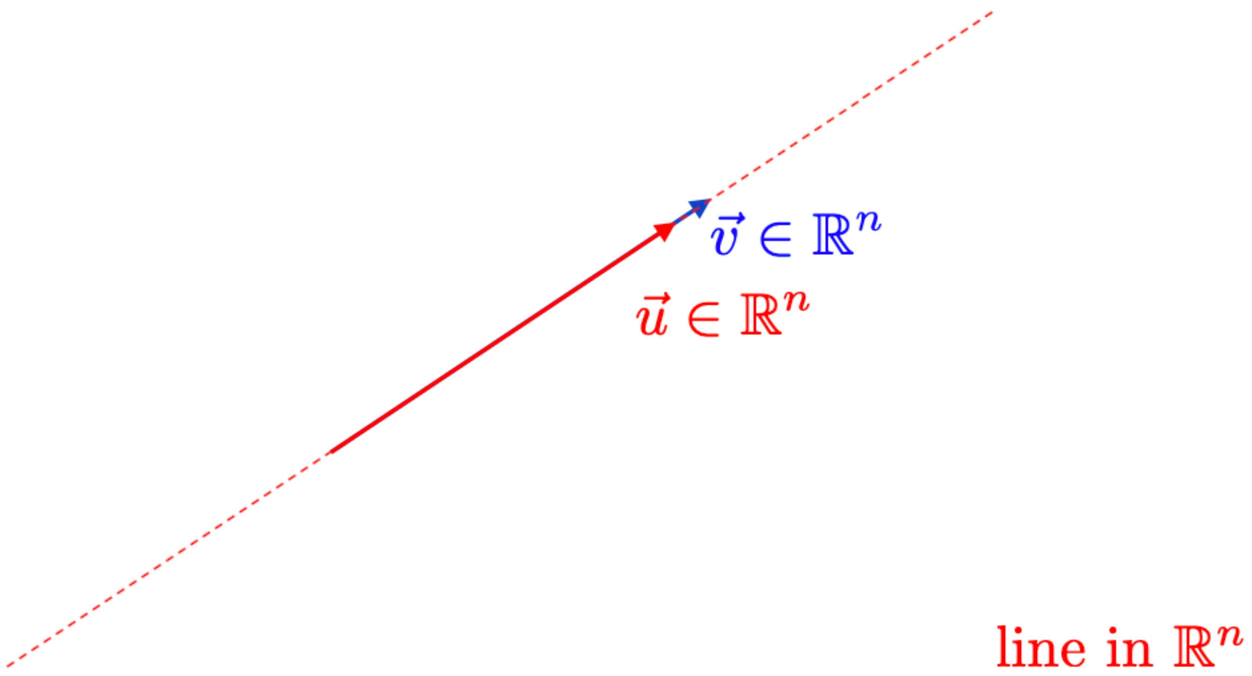
$$w = c_1v_1 + c_2v_2 + \cdots + c_pv_p$$

where $c_1, c_2, \ldots, c_p$ are scalars, $v_1, v_2, \ldots, v_p$ are vectors in $\mathbf{R}^n$, and $w$ is a vector in $\mathbf{R}^n$.

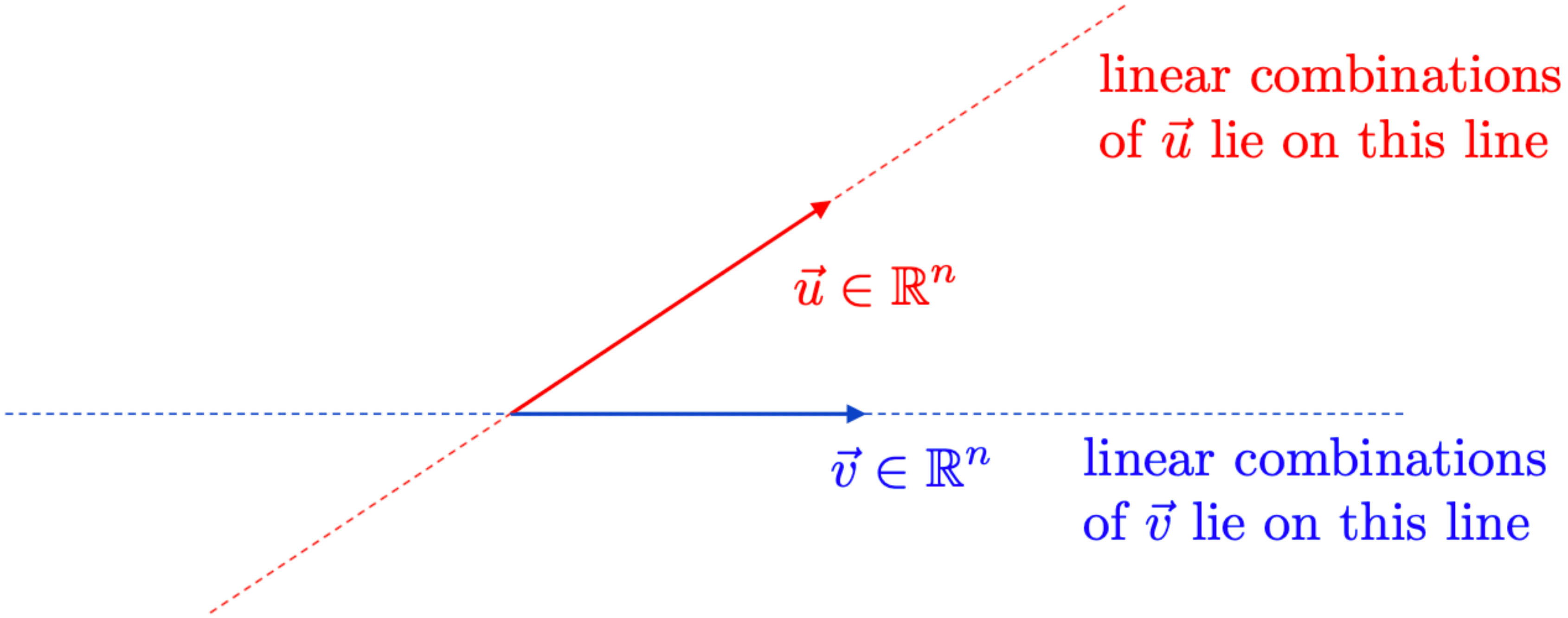
Definition

We call $w$ a **linear combination** of the vectors $v_1, v_2, \ldots, v_p$. The scalars $c_1, c_2, \ldots, c_p$ are called the **weights** or **coefficients**.

# Geometric Interpretation of Linear Combinations



# Geometric Interpretation of Linear Combinations




linear combinations of $\vec{u}$ and $\vec{v}$ lie on a plane in $\mathbb{R}^n$

# Transfer Linear Equation to a Linear Combination Problem

$$2x + y = 1$$
$$x + y = 1$$

# Spans

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text

# Reminder: Linear Combination

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# Span

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be a set of vectors in $\mathbb{R}^n$. We define

$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\} = \text{set of all linear combinations of } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$$

For example, what is the span of $(2, -4)$ and $(1, 1)$?

# Span

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be a set of vectors in $\mathbb{R}^n$. We define

$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\} = \text{set of all linear combinations of } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$$

Is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in the span of $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$?

# Span

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be a set of vectors in $\mathbb{R}^n$. We define

$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\} = \text{set of all linear combinations of } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$$

Is $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ in the span of $\begin{bmatrix} 2 \\ -4 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$?

# More Precise Definition

"the set of" "such that"

**Definition**

Let $v_1, v_2, \ldots, v_p$ be vectors in $\mathbf{R}^n$. The **span** of $v_1, v_2, \ldots, v_p$ is the collection of all linear combinations of $v_1, v_2, \ldots, v_p$, and is denoted $\text{Span}\{v_1, v_2, \ldots, v_p\}$. In symbols:

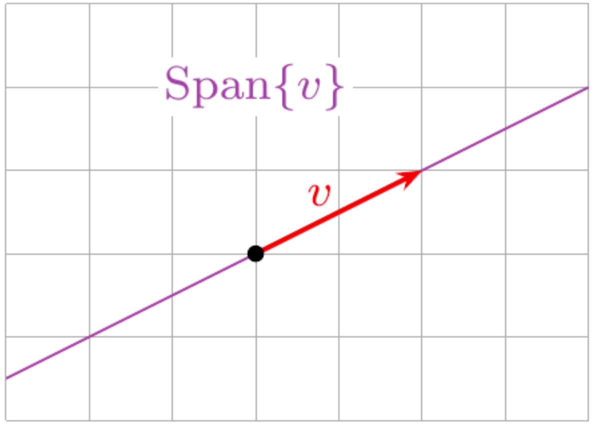
$$\text{Span}\{v_1, v_2, \ldots, v_p\} = \{ x_1 v_1 + x_2 v_2 + \cdots + x_p v_p \mid x_1, x_2, \ldots, x_p \text{ in } \mathbf{R} \}.$$

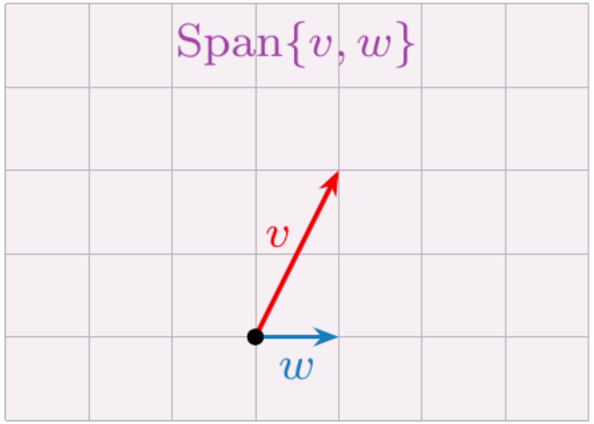
Synonyms: $\text{Span}\{v_1, v_2, \ldots, v_p\}$ is the subset **spanned by** or **generated by** $v_1, v_2, \ldots, v_p$.

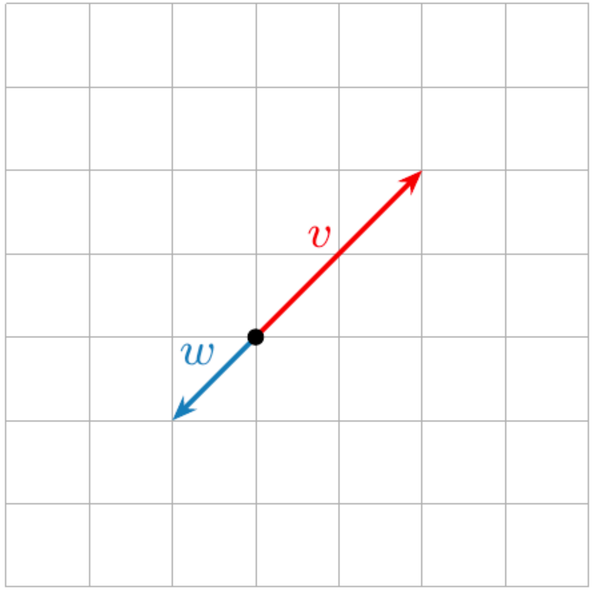
This is the first of several definitions in this class that you simply **must learn**. I will give you other ways to think about Span, and ways to draw pictures, but *this is the definition*. Having a vague idea what Span means will not help you solve any exam problems!

# Span in $\mathbb{R}^2$

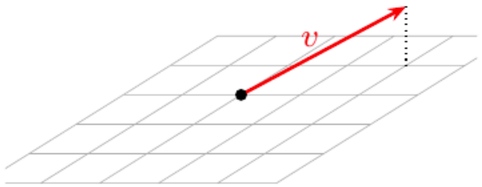
Drawing a picture of $\mathrm{Span}\{v_1, v_2, \ldots, v_p\}$ is the same as drawing a picture of all linear combinations of $v_1, v_2, \ldots, v_p$.



# Span in $\mathbb{R}^3$



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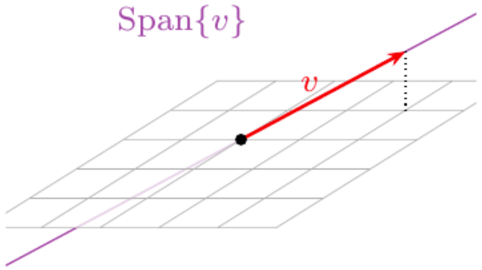

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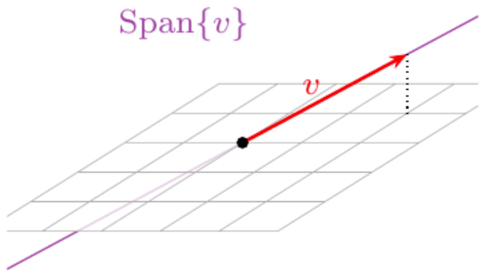


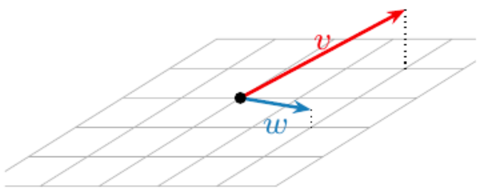

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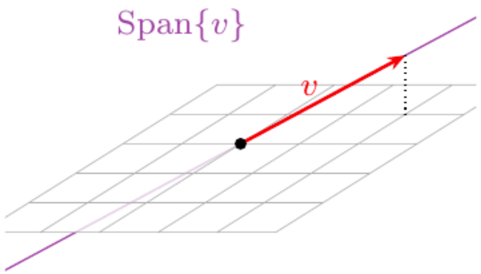


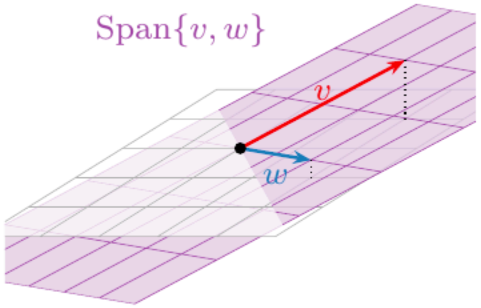

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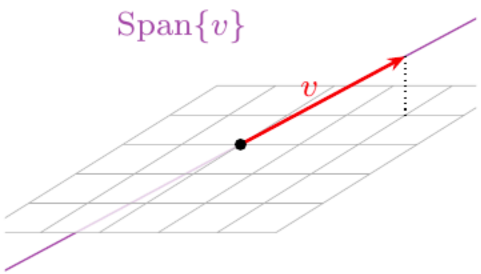


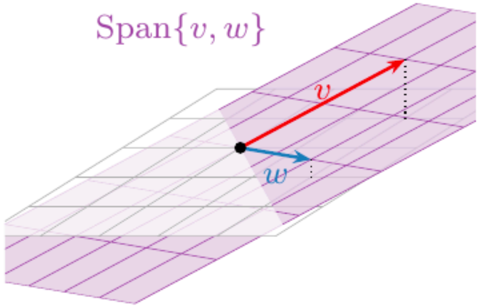


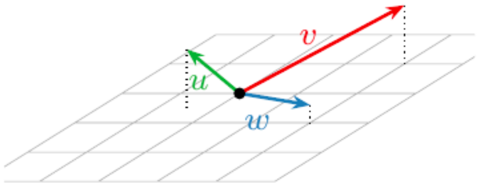

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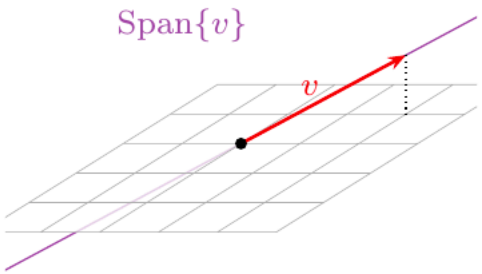


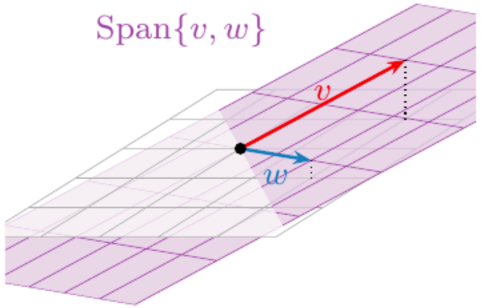


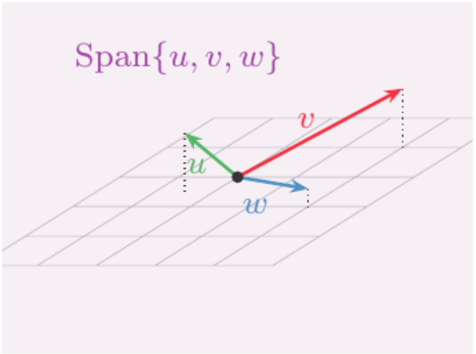

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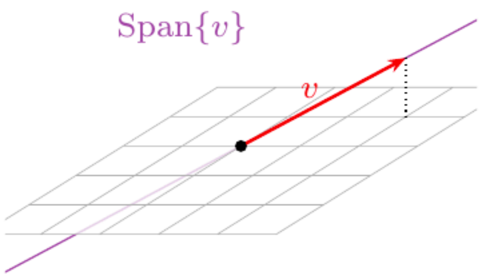


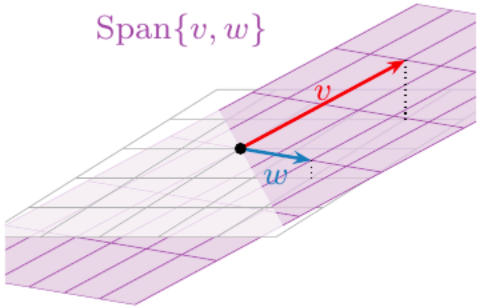


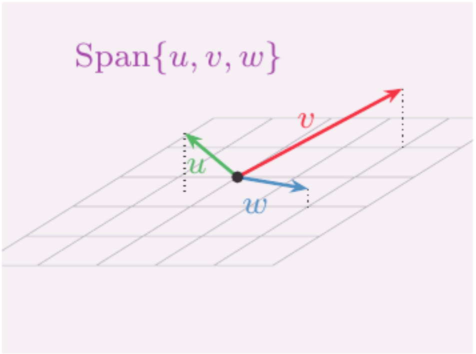


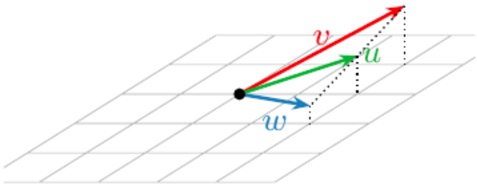

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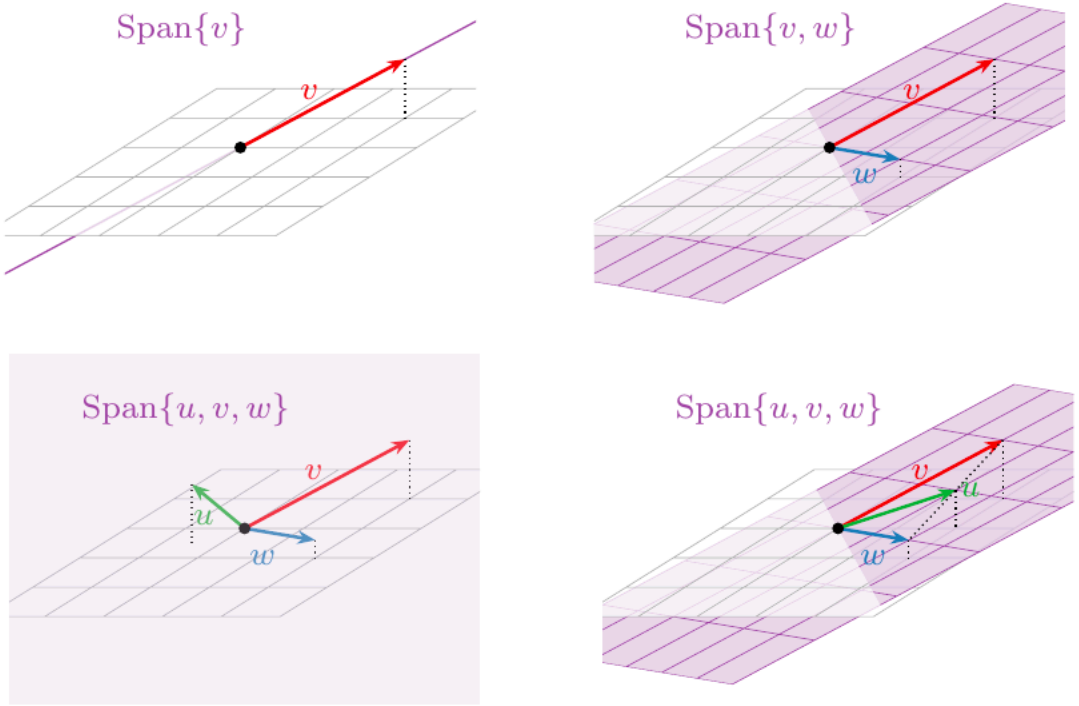

# Questions?