Lecture 15

Determinants

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Recap

- Orthogonal $q_i^T q_j = 0 \quad i \neq j$

  Orthonormal $q_i^T q_j = 0 \quad i \neq j \quad \|q_i\| = 1$

- $q_1 \cdots q_n$ orthonormal. which means $Q = [q_1 \cdots q_n]$

  Then $Q^T Q$ is identify

  - projection is as easy as. { coef is $Q^T b$; projection matrix is $Q Q^T$

Example. $q_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad q q^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

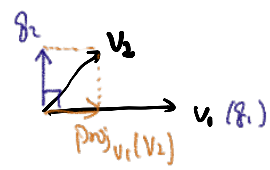
- if Q is a square matrix, all column vectors are orthonormal

  ⇓

  orthogonal matrix $\quad Q^T Q = Q Q^T = I. \quad Q^T = Q^{-1}$

Gram - Schmidt Process $\underbrace{\{v_1 \cdots v_n\}}_{A} \longrightarrow$ orthogonal basis $\{q_1 \cdots q_n\}$

- $q_1 = v_1$
- $q_2 = v_2 - \text{proj}_{\text{span}\{v_1\}}(v_2) = v_2 - \text{proj}_{\text{span}\{q_1\}}(v_2)$



- $q_3 = v_3 - \text{proj}_{\text{span}\{v_1, v_2\}}(v_3)$

  $= v_3 - \text{proj}_{\text{span}\{q_1, q_2\}}(v_3)$ easier to compute $q_1 . q_2$ are orthogonal.

1. (yellow) : ∃ QR decomposition. $A = QR$
2. (green) : It is easier to compute.

QR decomposition.

① using Gram Schmidt ((green)), to compute the Q.

② change each column of Q to unit vector (orthogonal → orthonormal)

③ $R = Q^T A$ because. $A = QR$. which means $Q^T A = \underbrace{Q^T Q}_{I} R = R$

Strang Sections 5.1 – Properties of Determinants

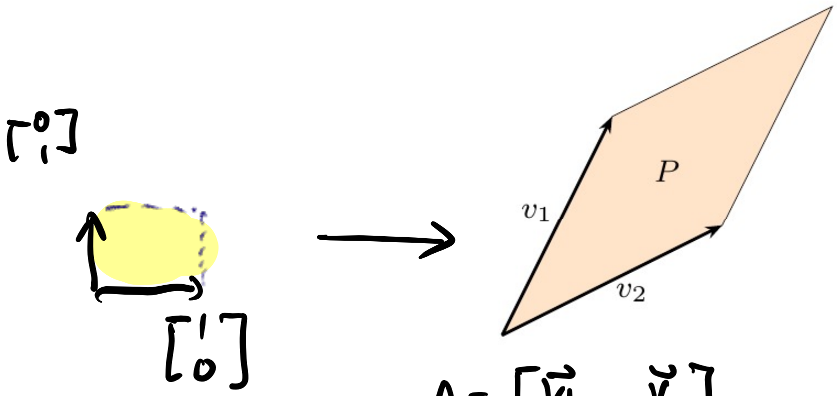
Course notes adapted from N. Hammoud's NYU lecture notes.

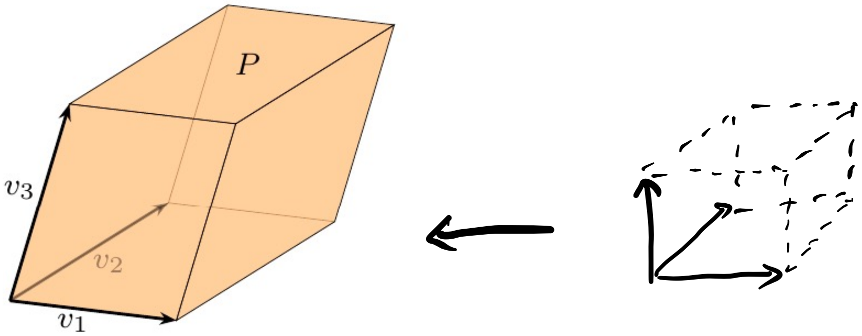
Introduction to Determinants

The Idea of Determinants

Let A be an $n \times n$ matrix. Determinants are only for square matrices.

The columns $v_1, v_2, \ldots, v_n$ give you n vectors in $\mathbf{R}^n$. These determine a **parallelepiped** P.





$A = [\vec{v}_1 \quad \vec{v}_2]$

$x \to Ax$

$A = [v_1, v_2, v_3]$

$x \to Ax$

lies in the same dimension

Determinants, as Volume Change

$\det = 0 \iff \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ are linear dependent.

$\iff A$ is not invertible.

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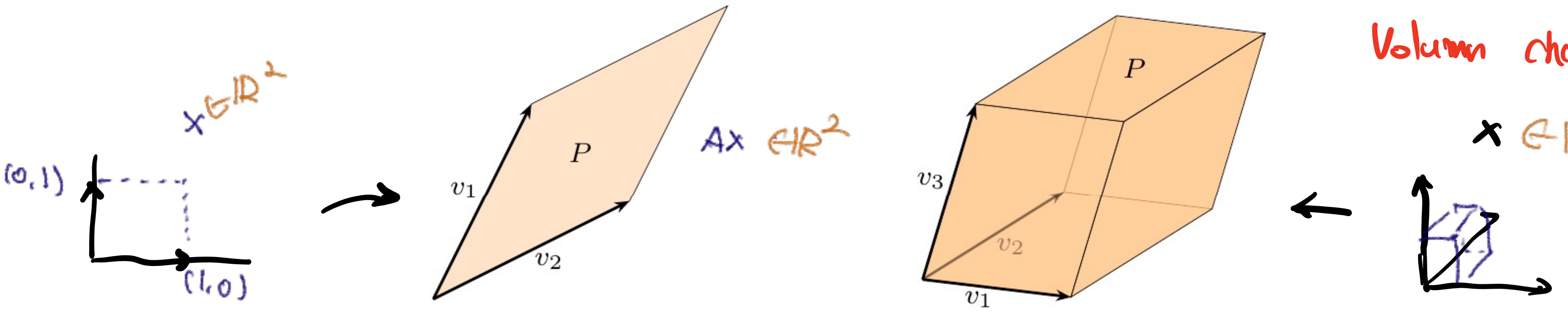
$Ax \in \mathbb{R}^3$ "det" is defined

Volumn change!



$A = [v_1, v_2]$ $A = [v_1, v_2, v_3]$

Want: $\det(A) = 0 \iff A$ is not invertible.

$\iff v_1 \ldots v_n$ are linear dependent.

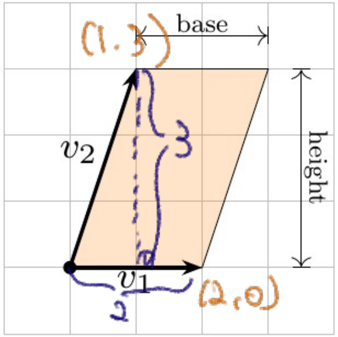
⇓

maps to something lower dimension than n

Determinants – 2 × 2 case

We already have a formula in the 2 × 2 case:

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$



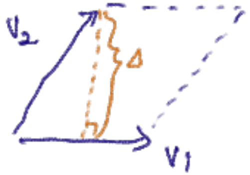
① $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0$ $ad = bc \Rightarrow \frac{a}{c} = \frac{b}{d} \Leftrightarrow \frac{a}{b} = \frac{c}{d}$

$\Leftrightarrow \begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix}$ are colinear

② Example

$A = [v_1 \; v_2] = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \Rightarrow 2 \times 3 - 1 \times 0 = 6$

$B = [v_2 \; v_1] = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} \Rightarrow 1 \times 0 - 2 \times 3 = -6$



$\Delta = v_2 - \mathrm{proj}_{v_1}(v_2)$

$\det([v_2 \; v_1]) = \|\Delta\| \cdot \|v_1\|$

★ $\det(A) = -\det(B)$!!

Determinants – 3 × 3 case

Here's the formula:

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

$a_{1f(1)}\, a_{2f(2)}\, a_{3f(3)}$

f(1), f(2), f(3) is different order of 1, 2, 3 "permutations"

f(1)	f(2)	f(3)	
1	2	3	+
2	3	1	+
3	1	2	+
3	2	1	−
1	3	2	−
2	1	3	−

$\pm\, a_{1f(1)}\, a_{2f(2)} \cdots a_{nf(n)}$

- f(1) ⋯ f(n) is the permutation of 1, ⋯ n
- (not required) If you start from 1, 2, ⋯, n each step you can swift two variable
  ① If you switch odd times it's −
  ② ⋯ even times it's +

ex. 1, 2, 3
→ 2, 1, 3 (2⟷1)
→ 3 2 1 (1⟷3)
→ 1, 3, 2 (2⟷3)

$$\det\begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix} =$$

Determinants – 3 × 3 case

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f(1)	f(2)	f(3)	
1	2	3	+
2	3	1	+
3	1	2	+
3	2	1	−
1	3	2	−
2	1	3	−

1↔3 → 321
2↔3 → ✓
1↔2 → 213
2↔3 → ✓

$a_{1f(1)} a_{2f(2)} \cdots a_{nf(n)}$

f(1) f(2) ⋯ f(n) is permutation of 1, 2, …, n

How to choose + or − (not required)

- Start from f(1)=1, f(2)=2 ⋯ f(n)=n
  each step you can switch two element
  1, 2, 3 — 2 1 3 (1↔2)
  1 3 2 (3↔2)
  3 2 1 (1↔3)
- If I need
  - odd time of switching −
  - even time of switching +.

$$\det\begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix} =$$

Determinants – n × n case

We can think of the determinant as a function of the entries of a matrix:

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{array}{c} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}. \end{array}$$

The formula for the determinant of an $n \times n$ matrix has $n!$ terms. So the determinant of a 10×10 matrix has 3,628,800 terms!

When mathematicians encounter a function whose formula is too difficult to write down, we try to *characterize* it in terms of its properties.

Determinants – Definition

Definition

The **determinant** is a function

$$\det\colon \mathbb{M}_{n\times n}(\mathbb{R}) \longrightarrow \mathbb{R}$$

with the following **defining properties**:

1. $\det(I_n) = 1$
2. If we do a row replacement, the determinant does not change.
3. If we swap two rows of a matrix, the determinant scales by -1.
4. If we scale a row of a matrix by k, the determinant scales by k.

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Why would we think of these properties? This is how volumes work!

1. The volume of the unit cube is 1.
2. Volumes don't change under a shear.
3. Volume of a mirror image is negative of the volume?
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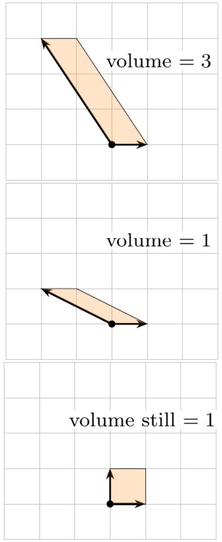
$$\det\begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} = 3$$

Scale: $R_2 = \frac{1}{3}R_2$

$$\det\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = 1$$

Row replacement: $R_1 = R_1 + 2R_2$

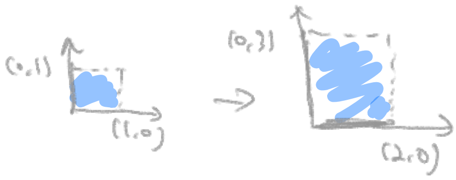
$$\det\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = 1$$



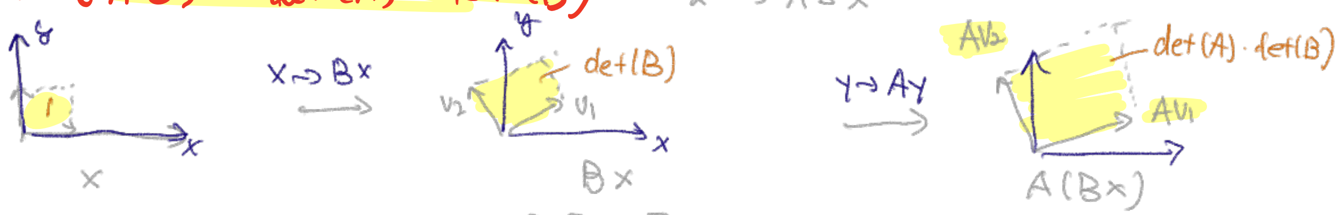
Properties of determinate

- det : Volume change. $\det([v_1, \cdots v_i \cdots v_j \cdots v_n]) = -(\det[v_1 \cdots v_j \cdots v_i \cdots v_n])$ ④ (switch the order.)
- $\det(I_n) = 1$ $x \to I_n x = x$ nothing changed.
- $\det(Q) = \pm 1$ Q: orthogonal
- $\det\left(\begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix}\right) = a_1 \cdots a_n$

Example $A = \begin{bmatrix} 2 & \\ & 3 \end{bmatrix}$ $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 3x_2 \end{pmatrix}$



- $\det(AB) = \det(A) \cdot \det(B)$ ① $x \to ABx$

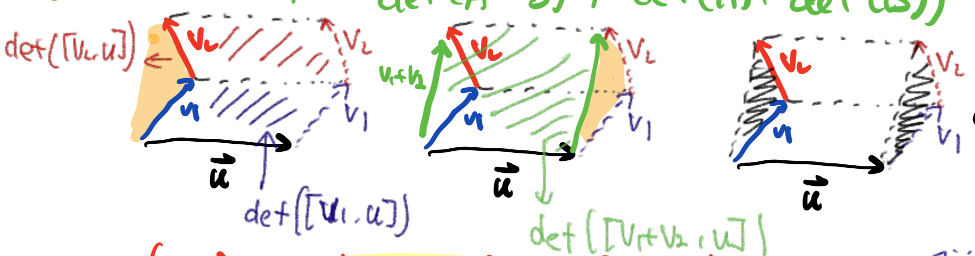


$\det(AB) = \det(A) \cdot \det(B)$

$AB = A[v_1\ v_2] = [Av_1\ Av_2]$ How we define matrix Multiplication

- $\det([\vec{v_1} + \vec{v_2}, \vec{u}]) = \det([\vec{v_1}, \vec{u}]) + \det([\vec{v_2}, \vec{u}])$ ② fix n-1 columns only L.C. in the rest Column

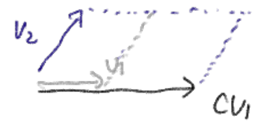
(Be careful $\det(A+B) \neq \det(A) + \det(B)$)



← in $\mathbb{R}^2$ plane!!!

$= \det([v_1, u]) + \det([v_2, u])$

- $\det([c\vec{v_1}, \vec{v_2}]) = c(\det[\vec{v_1}\ \vec{v_2}])$ ③



(Be careful $\det(cA) \neq c\det(A)$
$\det(cA) = c^n \det A$. all n columns multiply a scalar c (use ③ for n times)

Check. $\det([\vec{0}, \vec{v_1}, \cdots \vec{v}_{n-1}]) = 0$

$[\vec{0}, \vec{v_1} \cdots \vec{v}_{n-1}] = [0 \cdot \vec{a}, \vec{v_1} \cdots \vec{v}_{n-1}]$

by formula ③

$\Rightarrow \det([\vec{0}, \vec{v_1} \cdots \vec{v}_{n-1}]) = 0 \cdot \det([\vec{a}, \vec{v_1} \cdots \vec{v}_{n-1}])$
$= 0$

Properties of Determinants

Properties of Determinants

The determinant of an $n \times n$ matrix A is a number associated with A, and denoted by $\det A$ or $|A|$, with the following properties:

1. The determinant of the $n \times n$ identity matrix is 1.

2. The determinant changes sign when two rows are exchanged.

3. The determinant is a linear function of a fixed row.

- pull out constants: $\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = k \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

- break apart sums: $\begin{vmatrix} a + a' & b + b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

← Prop 2, 3, 4
are all right for rows

Attention!

$$\det(kA) \neq k \det A$$

$= k^n \det(A)$

$$\det(A + B) \neq \det A + \det B$$

Ex. $A = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix}$ $|A| = -5$

$B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ $|B| = 2$

$A + B = \begin{bmatrix} 2 & 1 \\ 2 & -2 \end{bmatrix}$ $|A+B| = -6$

Properties 1, 2 and 3

$c[v_1 \dots v_n] = [cv_1 \dots cv_n]$

1. The determinant of the $n \times n$ identity matrix is 1.

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   - pull out constants: $\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = k \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

   - break apart sums: $\begin{vmatrix} a + a' & b + b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

$\det[c_1 \vec{v}_1 , c_2 \vec{v}_2 , v_3]$ $\quad c_1 = -1. \quad c_2 = -1$

$= c_1 \cdot c_2 \det[v_1 , v_2 , v_3]$

Property 4

4. If A has two equal rows, then $\det A = 0$.

Colum

$A = [\vec{v_1}, \vec{v_1}, \vec{v_2}]$, $\det(A) = 0$ ⑤

④ $\det[v_1 \cdots v_i \cdots v_j \cdots v_n] = -\det[v_1 \cdots v_j \cdots v_i \cdots v_n]$

① $\det(AB) = \det(A)\det(B)$

② $\det[v_1 + v_1', u] = \det[v_1, u] + \det[v_1', u]$

③ $\det[cv_1, u] = c\det([v_1, u])$

proof. let's consider $B = [\vec{v_1}, -\vec{v_1}, \vec{v_2}]$

$\downarrow \times -1 \quad \downarrow \times -1$

$C = [-\vec{v_1}, \vec{v_1}, \vec{v_2}]$

first show $\det(B) = 0$

~ Hint

First show $\det(B) = -\det(C)$ by ④, switching $\vec{v_1}, -\vec{v_1}$

Secondly show $\det(B) = (-1) \times \det(C)$ by ③, Two columns times -1

by ③, second column times -1

$\Rightarrow \det(B) = 0 \Rightarrow \det(A) = -\det(B) = 0$

Property 5

5. The elementary row operation of adding $l \cdot (\text{row } i)$ to row j leaves the determinant unchanged.

$$\det([v_1 . v_2]) = \det([v_1 + cv_2 , v_2])$$

|| by ② . ③

$$\det[v_1 , v_2] + c \det [v_2 . v_2]$$

|| by ⑤ $\det[v_2 . v_2] = 0$

$$\det[v_1 , v_2]$$

(do elimination, the determinate

det (elimination matrix) =

Property 6

6. If A has a row of zeros, then $\det A = 0$.

Property 7

7. If A is triangular, then $\det A$ is the product of diagonal entries.

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & a_{nn} \end{vmatrix}$$

Property 8

8. A is invertible if and only if $\det A \neq 0$.

Property 9

9. $\det(AB) = \det A \cdot \det B$

Corollary – Determinant of the Inverse

Property 10

10. $\det A^T = \det A$

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