Lecture 15

Determinants

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Strang Sections 5.1 – Properties of Determinants

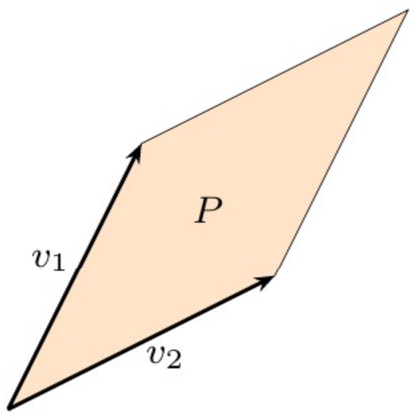
Course notes adapted from N. Hammoud's NYU lecture notes.

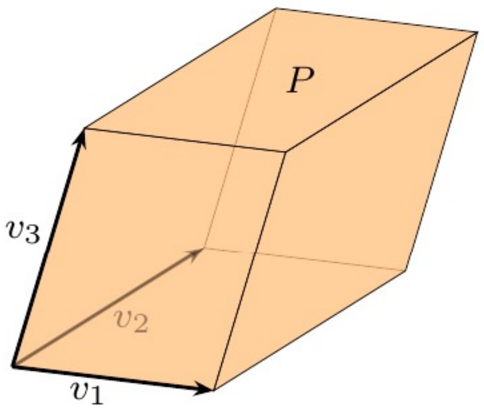
Introduction to Determinants

The Idea of Determinants

Let A be an $n \times n$ matrix. Determinants are only for square matrices.

The columns $v_1, v_2, \ldots, v_n$ give you n vectors in $\mathbf{R}^n$. These determine a **parallelepiped** P.

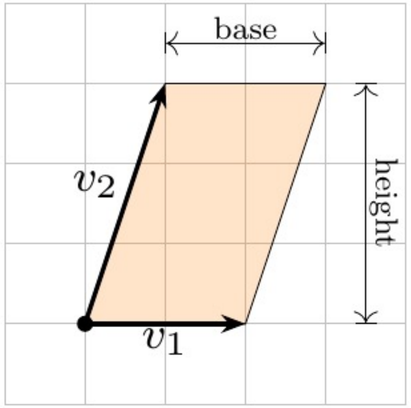




Determinants – 2 × 2 case

We already have a formula in the 2×2 case:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$



Determinants – 3 × 3 case

Here's the formula:

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

$$\det\begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix} =$$

Determinants – n × n case

We can think of the determinant as a function of the entries of a matrix:

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{aligned} &a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}. \end{aligned}$$

The formula for the determinant of an $n \times n$ matrix has $n!$ terms. So the determinant of a 10×10 matrix has 3,628,800 terms!

When mathematicians encounter a function whose formula is too difficult to write down, we try to *characterize* it in terms of its properties.

Determinants – Definition

Definition

The **determinant** is a function

$$\det\colon \mathbb{M}_{n\times n}(\mathbb{R}) \longrightarrow \mathbb{R}$$

with the following **defining properties**:

1. $\det(I_n) = 1$
2. If we do a row replacement, the determinant does not change.
3. If we swap two rows of a matrix, the determinant scales by -1.
4. If we scale a row of a matrix by k, the determinant scales by k.

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Why would we think of these properties? This is how volumes work!

1. The volume of the unit cube is 1.
2. Volumes don't change under a shear.
3. Volume of a mirror image is negative of the volume?
4. If you scale one coordinate by k, the volume is multiplied by k.

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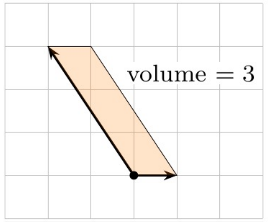
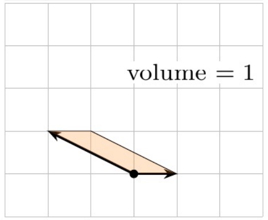
$$\det\begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} = 3$$



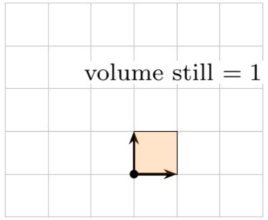
Scale: $R_2 = \frac{1}{3}R_2$

$$\det\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = 1$$



Row replacement: $R_1 = R_1 + 2R_2$

$$\det\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = 1$$



Properties of Determinants

Properties of Determinants

The determinant of an $n \times n$ matrix A is a number associated with A, and denoted by $\det A$ or $|A|$, with the following properties:

1. The determinant of the $n \times n$ identity matrix is 1.
2. The determinant changes sign when two rows are exchanged.
3. The determinant is a linear function of a fixed row.

- pull out constants: $\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = k \begin{vmatrix} a & b \\ c & d \end{vmatrix}$
- break apart sums: $\begin{vmatrix} a + a' & b + b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

Attention!

$$\det(kA) \neq k \det A$$

$$\det(A + B) \neq \det A + \det B$$

Properties 1, 2 and 3

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Property 4

4. If A has two equal rows, then $\det A = 0$.

Property 5

5. The elementary row operation of adding $l \cdot (\text{row}\, i)$ to row j leaves the determinant unchanged.

Property 6

6. If A has a row of zeros, then $\det A = 0$.

Property 7

7. If A is triangular, then $\det A$ is the product of diagonal entries.

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & a_{nn} \end{vmatrix}$$

Property 8

8. A is invertible if and only if $\det A \neq 0$.

Property 9

9. $\det(AB) = \det A \cdot \det B$

Corollary – Determinant of the Inverse

Property 10

10. $\det A^T = \det A$

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