NYU

Linear Algebra

Lecture 14

Orthogonal Bases

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Strang Sections 4.4 – Orthonormal Bases and Gram-Schmidt

Course notes adapted from N. Hammoud's NYU lecture notes.

Orthogonal Matrices

Orthogonal and Orthonormal Vectors

The vectors $\vec{q}_1, \ldots, \vec{q}_n$ are orthogonal if

$$\vec{q}_i \cdot \vec{q}_j = \vec{q}_i^T \vec{q}_j = 0 \qquad (i \neq j)$$

The vectors $\vec{q}_1, \ldots, \vec{q}_n$ are orthonormal if

$$\vec{q}_i^T \vec{q}_j = 0 \qquad (i \neq j)$$

$$||q_i|| = 1$$

Matrices with Orthonormal Columns

A matrix that has orthonormal columns is denoted by Q, where

$$Q^TQ = I$$

$$Q = [\vec{q}_1 \;\; \vec{q}_2 \ldots \vec{q}_n] \qquad \Longrightarrow \qquad Q^T = \begin{bmatrix} \vec{q}_1^T \\ \vec{q}_2^T \\ \vdots \\ \vec{q}_n^T \end{bmatrix}$$

Orthogonal Matrices

If Q is a square matrix with orthonormal columns, then Q is called an orthogonal matrix. In this case $Q^TQ = I$ and $QQ^T = I$.

Q is invertible with $Q^{-1} = Q^T$

Orthogonal and Orthonormal Bases

Orthogonal Bases

A set of vectors $\{\vec{q}_1, \dots, \vec{q}_n\}$ is called an orthogonal basis of a vector space V if $\vec{q}_1, \dots, \vec{q}_n$ are orthogonal and they span V.

Theorem : $\{\vec{q}_1, \dots, \vec{q}_n\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^m$, then $\vec{q}_1, \dots, \vec{q}_n$ are linearly independent and they form a basis for the subspace $S = \text{span}\{\vec{q}_1, \dots, \vec{q}_n\}$.

Theorem of Coefficients

Let $\{\vec{q}_1, \ldots, \vec{q}_n\}$ be an orthogonal basis for a subspace $S \subset \mathbb{R}^m$. For each $\vec{v} \in S$,

$$\vec{v} = c_1\vec{q}_1 + c_2\vec{q}_2 + \cdots + c_n\vec{q}_n$$

with $c_i = \dfrac{\vec{q}_i^{\,T}\vec{v}}{\vec{q}_i^{\,T}\vec{q}_i}$ for $1 \leq i \leq n$.

Gram-Schmidt

The Gram-Schmidt Process

Consider a vector space V with basis $\beta_V = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$

Gram-Schmidt (G-S) turns β_V into an orthogonal basis $\{\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n\}$ by using projections.

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$$
\begin{aligned}
\vec{q}_1 &= \vec{v}_1 \\
\vec{q}_2 &= \vec{v}_2 - \vec{p}_{21} = \vec{v}_2 - \frac{\vec{q}_1^{\,T}\vec{v}_2}{\vec{q}_1^{\,T}\vec{q}_1}\vec{q}_1 \\
\vec{q}_3 &= \vec{v}_3 - \vec{p}_{31} - \vec{p}_{32} \\
&= \vec{v}_3 - \frac{\vec{q}_1^{\,T}\vec{v}_3}{\vec{q}_1^{\,T}\vec{q}_1}\vec{q}_1 - \frac{\vec{q}_2^{\,T}\vec{v}_3}{\vec{q}_2^{\,T}\vec{q}_2}\vec{q}_2 \\
&\vdots \\
\vec{q}_n &= \vec{v}_n - \vec{p}_{n1} - \vec{p}_{n2} - \cdots - \vec{p}_{n(n-1)}
\end{aligned}
$$

Example

Another Example

Find an orthogonal basis $\{u_1, u_2, u_3\}$ for $W = \mathrm{Span}\{v_1, v_2, v_3\} = \mathbf{R}^3$, where

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}.$$

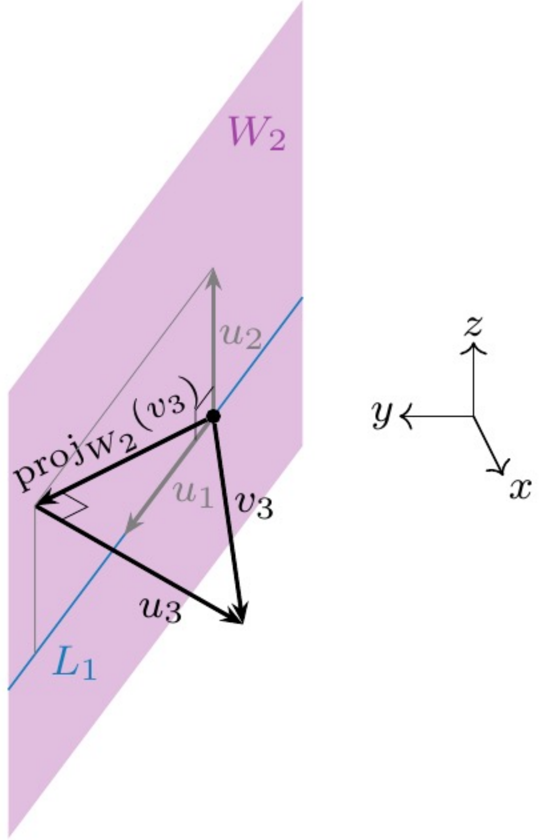
Example

Find an orthogonal basis $\{u_1, u_2, u_3\}$ for $W = \text{Span}\{v_1, v_2, v_3\} = \mathbf{R}^3$, where

$$v_1 = \begin{pmatrix}1\\1\\0\end{pmatrix},\ v_2 = \begin{pmatrix}1\\1\\1\end{pmatrix},\ v_3 = \begin{pmatrix}3\\1\\1\end{pmatrix} \overset{\text{G–S}}{\rightsquigarrow} u_1 = \begin{pmatrix}1\\1\\0\end{pmatrix},\ u_2 = \begin{pmatrix}0\\0\\1\end{pmatrix},\ u_3 = \begin{pmatrix}1\\-1\\0\end{pmatrix}$$

Why does this work?

- Once we have u_1 and u_2, then we're sad because v_3 is not orthogonal to u_1 and u_2.
- Fix: let $W_2 = \text{Span}\{u_1, u_2\}$, and let $u_3 = (v_3)_{W_2^\perp} = v_3 - \text{proj}_{W_3}(u_3)$.
- By construction, $u_1 \cdot u_3 = 0 = u_2 \cdot u_3$ because $W_2 \perp u_3$.



Check:

$$u_1 \cdot u_2 = 0 \ ✔$$
$$u_1 \cdot u_3 = 0 \ ✔$$
$$u_2 \cdot u_3 = 0 \ ✔$$

QR Factorization

The *QR* Factorization

Given an $m \times n$ matrix $A = [\vec{v}_1 \ \vec{v}_2 \ \ \dots \ \ \vec{v}_n]$, such that $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly independent. Then, we can factorize A as

$$A = QR$$

The *QR* Factorization

Given an $m \times n$ matrix $A = [\vec{v}_1\ \vec{v}_2\ \ldots\ \vec{v}_n]$, such that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly independent. Then, we can factorize A as

$$A = QR$$

Finding Q: Let $Q = [\vec{q}_1\ \vec{q}_2\ \ldots\ \vec{q}_n]$

To find $\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n$, we use G-S on the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, then make them orthonormal by dividing each $\vec{q}_i$ by its magnitude.

Finding R: $A = QR \implies$ multiply both sides by Q^T

$$\implies Q^T A = Q^T Q R \implies R = Q^T A$$

Example